University of Zagreb

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

Marija Galić

Analysis of a nonlinear three-dimensional problem of fluid, mesh and shell interaction

DOCTORAL THESIS

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Summary

In the first part of this thesis, we introduce the fluid-structure interaction problems that we are dealing with, and present a literature review of completed and ongoing research on this matter.

In the second part of this thesis we prove the existence of a weak solution to a linear fluid-structure interaction problem modeling the flow of an incompressible, viscous three-dimensional fluid, flowing through a cylinder whose lateral wall is described by the two-dimensional linearly elastic Koiter shell equations coupled with the one-dimensional elastic mesh equations. The fluid and the composite structure are fully coupled via the kinematic and dynamic coupling conditions describing continuity of velocity and balance of contact forces. The methodology of the proof is based on the semi-discretization approach, in which the full, coupled problem is discretized in time, and, at the same time, split into a fluid and a composite structure subproblem using the so-called Lie operator splitting strategy.

The third part of this thesis deals with a nonlinear, moving boundary fluid-structure interaction problem between an incompressible, viscous fluid flow and an elastic structure composed of a cylindrical shell supported by a mesh-like elastic structure. The first main difference with regards to the linear case is that the fluid flow, which is driven by the time-dependent dynamic pressure data, is modeled by the Navier-Stokes equations. Furthermore, we had to employ the Arbitrary-Lagrangian Eulerian mapping to deal with the motion of the fluid domain, which introduces an additional nonlinearities in the problem. Finally, we prove the existence of a weak solution to this nonlinear, moving boundary fluid-structure interaction problem by using the same strategy as in the linear case together with the non-trivial compactness results, which enabled us to pass to the limit in the weak formulation.

These problems were motivated by studying fluid-structure interaction between blood flow through coronary arteries treated with metallic mesh-like devices called stents.

Keywords: fluid-structure interaction; moving boundary problem; incompressible Navier-Stokes equations; linearly elastic Koiter shell; one-dimensional elastic mesh; kinematic coupling conditions; dynamic coupling conditions; operator splitting method; Arbitrary Lagrangian-Eulerian mapping; semi-discretization; weak solutions; compactness

Prošireni sažetak

U prvom dijelu ove radnje uvodimo probleme interakcije fluida i strukture kojima ćemo se baviti i dajemo pregled literature o istraživanjima koja su napravljena u tom području.

U drugom dijelu ove radnje dokazujemo egzistenciju slabog rješenja linearnog problema interakcije fluida i strukture koji modelira tok inkompresibilnog, viskoznog, trodimenzionalnog fluida kroz cilindričnu domenu čija lateralna granica je modelirana dvodimenzionalnim jednadžbama linearne elastične Koiterove ljuske spojenima s jednodimenzionalnim jednadžbama elastične mreže. Fluid i složena struktura su potpuno spojeni kinematičkim i dinamičkim rubnim uvjetima koji opisuju neprekidnost brzine i balans kontaktnih sila. Metodologija dokaza se sastoji u tome da naš puni problem podijelimo na dva jednostavnija potproblema, tj. potproblem za fluid i potproblem za strukturu, te ih semi-diskretiziramo, tj. diskretiziramo u vremenu, koristeći Lie operator splitting metodu.

Treći dio ove radnje se bavi nelinearnim problemom interakcije fluida i strukture s pomičnom granicom. Tok inkompresibilnog, viskoznog fluida je, za razliku od linearnog slučaja, modeliran Navier-Stokesovim jednadžbama i pokreće ga dinamički tlak na ulazu i izlazu iz cilindrične domene. Lateralna/pomična granica je opisana jednadžbama linearne elastične Koiterove ljuske spojenima s jednadžbama elastične mreže i na njoj su zadani kinematički i dinamički uvjeti spajanja. S obzirom da je lateralna granica pomična, u svakom trenutku je njen položaj jedna od nepoznanica problema, što uvodi dodatne nelinearnosti u problem. Dokazujemo egzistenciju slabog rješenja ovog nelinearnog problema koristeći Lie operator splitting metodu i semi-diskretizaciju, kao i u linearnom slučaju, Arbitrary Lagrangian-Eulerian preslikavanje kojim "fiksiramo" pomičnu granicu, te netrivijalne rezultate kompaktnosti koji nam omogućuju prelazak na limes u slaboj formulaciji.

Motivacija za oba problema dolazi iz primjena u hemodinamici, odnosno iz proučavanja toka krvi kroz koronarne arterije tretirane sa stentovima. Stent je mala, metalna, mrežasta cjevčica koja se postavlja u suženi ili zatvoreni dio koronarne arterije s ciljem otvaranja i uspostavljanja normalnog protoka krvi.

Ključne riječi: interakcija fluida i strukture; problem pomične granice; inkompresibilne Navier-Stokesove jednadžbe; linearna elastična Koiterova ljuska; jednodimenzionalna elastična mreža; kinematički uvjeti spajanja; dinamički uvjeti spajanja; *operator splitting*

metoda; *Arbitrary Lagrangian-Eulerian* preslikavanje; semi-diskretizacija; slaba rješenja; kompaktnost

Contents

Chapter 1

Introduction

1.1 Problem description

We study a fluid-structure interaction (FSI) problem between the flow of a viscous, incompressible, Newtonian fluid in a 3d cylindrical domain, and an elastic, composite structure, consisting of a cylindrical shell supported by a mesh-like elastic structure.

The linear Koiter shell equations allowing displacement in all three spatial directions are used to model the elastodynamics of the lateral wall of the fluid domain, and a 1d hyperbolic net model consisting of a collection of linearly elastic curved rods is used to model the elastodynamics of the mesh-like structure. The mesh and the shell are coupled via the no-slip condition and via the balance of contact forces and moments. The resulting composite structure is then coupled to the fluid equations through the kinematic and dynamic coupling conditions, describing continuity of velocity and balance of forces which are evaluated along the fluid-structure interface.

Considering the fluid flow, two cases are considered. In the first case, the fluid flow is assumed to be laminar, modeled by the time-dependent Stokes equations, and the shell displacement is assumed to be small giving rise to a fixed fluid-structure interface, and consequently to a linear fluid-structure interaction problem.

In the second case, the fluid flow, which is driven by the time-dependent dynamic pressure data, is modeled by the Navier-Stokes equations. The lateral fluid domain boundary, whose elastic properties are described by the above mentioned composite structure, depends on the motion of the fluid occupying the domain and is one of the unknowns of the problem. The moving-boundary, together with the nonlinear coupling of the fluid and the composite structure along the fluid-structure interface, leads to a highly nontrivial fluid-structure interaction problem.

Problems of this type arise in many applications. In particular, the problems studied here were motivated by a study of blood flow through coronary arteries treated with vascular stents. A stent is a metallic mesh-like device, which is inserted into a clogged artery in order to keep the passageway open. A better understanding of the complex interaction between blood flow, artery and stent can lead to improved stent design (see e.g. [16]).

1.2 Literature overview

The development of existence theory for moving-boundary, fluid-structure interaction problems, has become particularly active since the late 1990's. The first existence results were obtained for the cases in which the structure is completely immersed in the fluid, and the structure was considered to be either a rigid body, or described by a finite number of modal functions, see for example [10, 34, 37, 38, 40, 41, 42].

More recently, the coupling between the two-dimensional or three-dimensional Navier-Stokes equations an two-dimensional or three-dimensional linear elasticity on fixed domains, was considered for linear models in [3, 4, 39], and for nonlinear models in [6, 7, 26, 27, 28, 54].

Concerning compliant (elastic or viscoelastic) structures, the first FSI existence result, locally in time was obtained in [8], where a strong solution for an interaction between an incompressible, viscous two-dimensional fluid and a one dimensional viscoelastic string was obtained assuming periodic boundary conditions. This result was extended in [56], where the existence of a unique, local in time, strong solution for any data, and the existence of a global strong solution for small data, were proved in the case when the structure was modeled as a clamped viscoelastic beam. Coutand and Shkoller proved the existence, locally in time, of a unique regular solution between a three-dimensional incompressible, viscous fluid, and a three-dimensional structure, immersed in fluid, where the structure was modeled by the equations of linear [35], or quasi-linear [36] elasticity. In the case when the structure is modeled by a linear wave equation, Kukavica and Tufahha proved the existence, locally in time, of a strong solution, assuming lower regularity for the initial data [53]. A fluid-structure interaction between a three-dimensional viscous, incompressible fluid and two-dimensional elastic shells was considered in [24, 25], where local-in-time existence of the unique regular solution was proved.

In the context of weak solutions, the following results have been obtained. In [23], existence of weak solutions for the unsteady interaction of a three-dimensional incompressible, viscous fluid, and a two-dimensional viscoelastic plate was considered. Grandmont improved this result to hold for a two-dimensional elastic plate in [45]. The aforementioned author, together with Hillariet, proved the existence of global strong solutions to a beam-fluid interaction system in [46].

In [57], the authors proved existence of weak solutions to a class of FSI problems modeling the flow of an incompressible, viscous, Newtonian fluid flowing through a cylinder whose lateral wall was modeled by either the linearly viscoelastic, or by the linearly elastic Koiter shell equations, assuming nonlinear coupling at the deformed fluid-structure interface. Similar problem was considered in [55]. In [61], the authors considered the fluid-structure interaction with Navier slip boundary condition at the interface, and in contrast with other papers, they considered structure that has both radial and tangential displacements non-negligible. The three-dimensional geometry was first considered in [58], but only with the radial displacement of the structure. A weak solution to the FSI problem where structure is modeled as an elastic shell with nonlinear membrane energy containing an additional regularizing term was constructed in [60]. For the completeness of this literature overview, we also enlist some papers that include numerical simulations [5, 9, 13, 49].

None of the works mentioned above, however, considered a composite structure. The only works in which analysis of an FSI problem including an approximation of a stent-supported vessel were considered are [12], [22] and [59]. In [22] a different, simplified, reduced coupled problems was studied, and in both papers the presence of a stent was modeled by the jump in the elasticity coefficients of a shell. In [12] a Koiter shell allowing only radial displacement was considered. This is significantly different from the present work where a stent is modeled as a separate mesh-like structure, and displacement in all three spatial directions is taken into account.

Until recently, mathematical modeling of stents has been based almost exclusively on $3D$ approaches where a stent is assumed to be a single, $3D$ elastic body, approximated using $3D$-based finite elements. Such approaches are associated with large computer memory requirements and significant computing time due to the slender nature of the local stent components, known as stent struts. To reduce computational costs, an alternative approach was suggested in [64], where a stent was modeled as a network of $1D$ curved rods, allowing the use of $1D$ finite elements for their numerical simulation. Although the model is one-dimensional, it provides $3D$ information about the deformation of stent struts in all three spatial directions. The resulting model has been justified both computationally [19] and mathematically [47, 50, 51].

In a recent work [18], this $1D$ stent net problem was coupled to an elastic shell of Naghdi type, and the corresponding *static* problem was analyzed. No fluid was considered. This contrasts the present work where *dynamic* models for both the stent and shell are considered, the shell is modeled using the cylindrical Koiter shell equations, and the resulting composite structure is coupled to the motion of an incompressible, viscous fluid.

In this thesis we prove the existence of weak solutions to two FSI problems modeling the flow of an incompressible, viscous, Newtonian fluid flowing through a cylinder whose lateral wall is described by the linearly elastic Koiter shell equations coupled with the linearly elastic $1D$ curved rod equations. In both cases, i.e. linear and nonlinear, the methodology of the proof is based on the same approach. We use the time-discretiation via Lie operator splitting to divide our problems into two subproblems with different physical properties. This method was used in [48] for a design of a stable, loosely coupled numerical scheme, called the kinematically coupled scheme (see also [15]). Crucial for a design of a stable scheme (stability was proved in [20]) is the inclusion of structure inertia into the fluid subproblem, which guarantees energy balance at the time discrete level, thereby avoiding stability problems due to the so called added mass effect [21]. Added mass effect is used to describe the elastodynamics of structures interacting with fluids with comparable densities, for which there is a significant exchange of energy between the fluid and structure motion, potentially causing instabilities is schemes that do not approximate well the energy exchange that occurs at the continuous level. These problems were also addressed in [14] and [59]. The Lie operator splitting scheme has been widely used in numerical computations, see [44] and references therein.

In the nonlinear, moving boundary problem, we had to employ Arbitrary Lagrangian-Eulerian mapping in order to overcome the difficulties that arise due to the motion of the fluid domain boundary. Furthermore, we had to employ a version of Aubin-Lions-Simon lemma, proven in [62], to be able to pass to the limit.

1.3 Chapter overview

The second chapter titled *Linear fluid-mesh-shell interaction problem* deals with the proof of the existence of a weak solution to a linear fluid-structure interaction problem modeling the flow of an incompressible, viscous three-dimensional fluid, flowing through a cylinder whose lateral wall is described by the two-dimensional linearly elastic Koiter shell equations coupled with the one-dimensional elastic mesh equations. The fluid and the composite structure are fully coupled via the kinematic and dynamic coupling conditions describing continuity of velocity and balance of contact forces. The methodology of the proof is based on the semi-discretization approach, in which the full, coupled problem is discretized in time, and, at the same time, split into a fluid and a composite structure subproblem using the so-called Lie operator splitting strategy. The results obtained in this chapter can be found in the article written by the author of this thesis and collaborators [17], and which is currently under revision.

The third chapter titled *Nonlinear, moving boundary fluid-mesh-shell interaction problem* deals with a nonlinear, moving boundary fluid-structure interaction problem between an incompressible, viscous fluid flow and an elastic structure composed of a cylindrical shell supported by a mesh-like elastic structure. The first main difference with regards to the linear case is that the fluid flow, which is driven by the time-dependent dynamic pressure data, is modeled by the Navier-Stokes equations. Furthermore, we had to employ the Arbitrary-Lagrangian Eulerian mapping to deal with the motion of the fluid domain, which introduces an additional nonlinearities in the problem. Finally, we prove the existence of a weak solution to this nonlinear, moving boundary fluid-structure interaction problem by using the same strategy as in the linear case together with the non-trivial compactness results, which enables us to pass to the limit in the weak formulation.

Chapter 2

Linear fluid-mesh-shell interaction problem

2.1 Model description

2.1.1 The fluid

We consider the flow of an incompressible, viscous fluid through a cylindrical domain, denoted by Ω:

$$\Omega = \{(z, x, y) \in \mathbb{R}^3 : z \in (0, L), \sqrt{x^2 + y^2} \leq R\}.$$

The fluid domain boundary consists of three parts: the lateral boundary Γ, which is a cylinder of radius R, the inlet boundary Γ_{in}, which is a circular area of radius R located at $z = 0$, and the outlet boundary Γ_{out}, which is a circular area of radius R, located at $z = L$.

The time-dependent Stokes equations for an incompressible, viscous fluid are used to model the flow in Ω:

$$\left.\begin{array}{rcl} \rho_F \partial_t \mathbf{u} & = & \nabla \cdot \boldsymbol{\sigma} \\ \nabla \cdot \mathbf{u} & = & 0 \end{array}\right\} \text{ in } \Omega, \quad t \in (0, T), \tag{2.1}$$

where ρ_F denotes the fluid density, $\mathbf{u}$ is the fluid velocity, $\boldsymbol{\sigma} = -pI + 2\mu_F \mathbf{D}(\mathbf{u})$ is the fluid Cauchy stress tensor, p is the fluid pressure, μ_F is the dynamic viscosity coefficient, and $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$. At the inlet and outlet we prescribe the pressure, with the tangential fluid velocity equal to zero (see [33]):

$$\left.\begin{array}{c} p = P_{in/out}(t) \\ \mathbf{u} \times \mathbf{e}_z = 0 \end{array}\right\} \text{ on } \Gamma_{in/out},$$

where $P_{in/out}$ are given. Therefore, the fluid flow is driven by the pressure drop, and the fluid flow is orthogonal to the inlet and outlet boundary.

The fluid velocity will be assumed to belong to the following classical function space

$$V_F = \{\mathbf{u} \in H^1(\Omega;\mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \times \mathbf{e}_z = 0 \text{ on } \Gamma_{in/out}\}. \tag{2.2}$$

2.1.2 The shell

The lateral boundary of the fluid domain will be assumed elastic, and modeled by the cylindrical Koiter shell equations. The shell thickness will be denoted by $h > 0$, the length by L, and its reference radius of the middle surface by R. We consider a clamped cylindrical shell. This reference configuration, which we denote by Γ, can be parameterized by

$$\boldsymbol{\varphi} : \omega \to \mathbb{R}^3, \quad \boldsymbol{\varphi}(z,\theta) = (R\cos\theta, R\sin\theta, z),$$

where $\omega = (0, L) \times (0, 2\pi)$, and $R > 0$, thus:

$$\Gamma = \{(R\cos\theta, R\sin\theta, z) : z \in (0, L), \theta \in (0, 2\pi)\}.$$

The first fundamental form of the cylinder Γ, also known as the metric tensor, will be denoted by A_c in covariant components, and by A_c in contravariant components:

$$A_c = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \quad A^c = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix},$$

and the area element is $dS = \sqrt{\det A_c}\ dzd\theta = R\ dzd\theta$. The second fundamental form of the cylinder Γ, or the curvature tensor in covariant components is given by

$$B_c = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}.$$

Under loading, the Koiter shell is displaced from its reference configuration Γ by a displacement $\boldsymbol{\eta} = \boldsymbol{\eta}(t, z, \theta) = (\eta_z, \eta_r, \eta_\theta)$, where η_z, η_r, and η_θ denote the tangential, radial and azimuthal components of displacement. The end points of the shell will be assumed to be clamped, giving rise to the following boundary conditions:

$$\begin{aligned} \boldsymbol{\eta}(t, 0, \theta) = \boldsymbol{\eta}(t, L, \theta) = 0, \theta \in (0, 2\pi), \\ \partial_z \eta_r(t, 0, \theta) = \partial_z \eta_r(t, L, \theta) = 0, \theta \in (0, 2\pi), \end{aligned}$$

whereas the boundary conditions at $\theta = 0, 2\pi$, will be periodic:

$$\begin{aligned} \boldsymbol{\eta}(t, z, 0) = \boldsymbol{\eta}(t, z, 2\pi), z \in (0, L), \\ \partial_\theta \eta_r(t, z, 0) = \partial_\theta \eta_r(t, z, 2\pi), z \in (0, L). \end{aligned}$$

The elastic properties of the shell are defined by the following elasticity tensor $\mathcal{A}$:

$$\mathcal{A}E = \frac{2\lambda\mu}{\lambda+2\mu}(A^c \cdot E)A^c + 2\mu A^c E A^c, \quad E \in \mathrm{Sym}(\mathbb{R}^2),$$

where λ and μ are Lamé constants. Tensor $\mathcal{A}$ defines the following elastic energy of the deformed Koiter shell:

$$E(\boldsymbol{\eta}) = \frac{h}{2}\int_\omega \mathcal{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\eta})R + \frac{h^3}{24}\int_\omega \mathcal{A}\boldsymbol{\varrho}(\boldsymbol{\eta}) : \boldsymbol{\varrho}(\boldsymbol{\eta})R, \tag{2.3}$$

where $\boldsymbol{\gamma}$ denotes the linearized change of metric tensor, measuring the stretch of the middle surface (membrane effects), and $\boldsymbol{\varrho}$ denotes the linearized change of curvature tensor, measuring flexure (bending, shell effects). They are given by:

$$\boldsymbol{\gamma}(\boldsymbol{\eta}) = \begin{pmatrix} \partial_z \eta_z & \frac{1}{2}(\partial_\theta \eta_z + R\partial_z \eta_\theta) \\ \frac{1}{2}(\partial_\theta \eta_z + R\partial_z \eta_\theta) & R\partial_\theta \eta_\theta + R\eta_r \end{pmatrix},$$

$$\boldsymbol{\varrho}(\boldsymbol{\eta}) = \begin{pmatrix} -\partial_{zz}\eta_r & -\partial_{z\theta}\eta_r + \partial_z \eta_\theta \\ -\partial_{z\theta}\eta_r + \partial_z \eta_\theta & -\partial_{\theta\theta}\eta_r + 2\partial_\theta \eta_\theta + \eta_r \end{pmatrix}.$$

Let V_K denote the following function space:

$$\begin{aligned} V_K = \{\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in H^1(\omega) \times H^2(\omega) \times H^1(\omega) :& \\ \boldsymbol{\eta}(t,z,\theta) = \partial_z \eta_r(t,z,\theta) = 0, z \in \{0, L\}, \theta \in (0, 2\pi),& \\ \boldsymbol{\eta}(t,z,0) = \boldsymbol{\eta}(t,z,2\pi), \partial_\theta \eta_r(t,z,0) = \partial_\theta \eta_r(t,z,2\pi), z \in (0,L)\},& \end{aligned} \tag{2.4}$$

equipped with the corresponding norm:

$$\|\boldsymbol{\eta}\|_k^2 = \|\eta_z\|^2_{H^1(\omega)} + \|\eta_r\|^2_{H^2(\omega)} + \|\eta_\theta\|^2_{H^1(\omega)}.$$

We are interested in weak solutions $\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in V_K$ satisfying the following elastodynamics problem for a cylindrical Koiter shell (see [31],[52]): find $\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in V_K$ such that

$$\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \boldsymbol{\mathcal{L}}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle = \int_\omega \mathbf{f} \cdot \boldsymbol{\psi} R, \quad \forall \boldsymbol{\psi} \in V_K, \tag{2.5}$$

where ρ_K is the shell density, $\mathbf{f}$ is the outside loading, and $\boldsymbol{\mathcal{L}}$ is the linear operator associated with the Koiter elastic energy (2.3):

$$\langle \boldsymbol{\mathcal{L}}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle = h\int_\omega \mathcal{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\psi})R + \frac{h^3}{12}\int_\omega \mathcal{A}\boldsymbol{\varrho}(\boldsymbol{\eta}) : \boldsymbol{\varrho}(\boldsymbol{\psi})R.$$

We emphasize that from Theorem 2.6-4 in [30], we get the coercivity of the operator $\boldsymbol{\mathcal{L}}$, i.e. $\langle \boldsymbol{\mathcal{L}}\boldsymbol{\eta}, \boldsymbol{\eta}\rangle \geq c\|\boldsymbol{\eta}\|_k^2$, $\forall \boldsymbol{\eta} \in V_K$. The differential form of the cylindrical Koiter shell elastodynamics problem on $(0,T)\times\omega$ is then given by:

$$\rho_K h\partial_t^2\boldsymbol{\eta}R+\mathcal{L}\boldsymbol{\eta}=\mathbf{f}R, \tag{2.6}$$

where $\mathbf{f}$ is outside force density, and $\mathcal{L}$ corresponds to the elastic force associated with the elastic energy (2.3).

2.1.3 The elastic mesh

An elastic mesh is a three-dimensional elastic body defined as a union of three-dimensional slender components called struts [64, 19]. Since struts are slender or "thin", meaning that the ratio between the thickness of each strut versus its length is small, 1D (reduced) models can be used to approximate their elastodynamic properties. In particular, keeping the stent application in mind, we will be using a 1D curved rod model to approximate the elastodynamic properties of slender mesh struts. The one space dimension corresponds to the parameterization of the middle line of curved rod. For the i-th curved rod, the middle line is parameterized via

$$\mathbf{P}_i:[0,l_i]\to\boldsymbol{\varphi}(\overline{\omega}),\quad i=1,\dots,n_E,$$

where n_E denotes the number of curved rods in a mesh. By using $s\in(0,l_i)$ to denote the location along the middle line, and $\mathbf{d}_i(t,s)$ to denote the displacement of the middle line from its reference configuration, $\mathbf{w}_i(t,s)$ the infinitesimal rotation of cross-sections, $\mathbf{q}_i(t,s)$ the contact moment, and $\mathbf{p}_i(t,s)$ the contact force, the following system of equations will be used to model the elastodynamics of 1D curved rods:

$$\begin{aligned}\rho_S A_i\partial_t^2\mathbf{d}_i &= \partial_s\mathbf{p}_i+\mathbf{f}_i,\\ \rho_S M_i\partial_t^2\mathbf{w}_i &= \partial_s\mathbf{q}_i+\mathbf{t}_i\times\mathbf{p}_i,\\ 0 &= \partial_s\mathbf{w}_i-Q_iH_i^{-1}Q_i^T\mathbf{q}_i,\\ 0 &= \partial_s\mathbf{d}_i+\mathbf{t}_i\times\mathbf{w}_i.\end{aligned} \tag{2.7}$$

Here, ρ_S is the strut's material density, A_i is the cross-sectional area of the $i-$th rod, M_i is the matrix related to the moments of inertia of the cross-sections, $\mathbf{f}_i$ is the line force density acting on the $i-$th rod, and $\mathbf{t}_i$ is the unit tangent on the middle line of the $i-$th rod. Matrix H_i is a positive definite matrix which describes the elastic properties and the geometry of the cross section, while matrix $Q_i\in SO(3)$ represents the local basis at each point of the middle line of the $i-$th rod (see [2] for more details). The first two equations describe the linear impulse-momentum law and the angular impulse-momentum law, respectively, while the last two equations describe a constitutive relation for a curved, linearly elastic rod, and the condition of inextensibility and unshearability, respectively.

System (2.7) is defined on a graph domain, determined by the geometry and topology of the mesh structure. The graph consists of a set of vertices $\mathcal{V}$ (points where the middle lines meet), and a set of edges $\mathcal{E}$ (pairing of vertices). The ordered pair $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ defines the mesh net topology. At each vertex $V \in \mathcal{V}$, the following coupling conditions are enforced:

- kinematic coupling conditions describing continuity of displacements and infinitesimal rotations,
- dynamic coupling conditions describing the balance of contact forces and contact moments.

Here we note that even though each individual mesh strut is inextensible, the mesh as whole can stretch/squeeze because the position of vertices is not fixed.

We are interested in weak solutions to the 1D mesh net problem, i.e., to the problem consisting of all the functions satisfying the system of linear equations (2.7) on a graph domain, with the above-mentioned coupling conditions holding at graph's vertices. To define the weak solution space, we first introduce a function space consisting of all the H^1-functions $(\mathbf{d}, \mathbf{w})$ defined on the entire net $\mathcal{N}$, such that they satisfy the kinematic coupling conditions at each vertex $V \in \mathcal{V}$:

$$\begin{aligned} H^1(\mathcal{N}; \mathbb{R}^6) = \{(\mathbf{d}, \mathbf{w}) = ((\mathbf{d}_1, \mathbf{w}_1), \dots, (\mathbf{d}_{n_E}, \mathbf{w}_{n_E})) \in \prod_{i=1}^{n_E} H^1(0, l_i; \mathbb{R}^6) :& \\ \mathbf{d}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{d}_j(\mathbf{P}_j^{-1}(V)), \mathbf{w}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{w}_j(\mathbf{P}_j^{-1}(V)),& \\ \forall V \in \mathcal{V}, V = e_i \cap e_j,\ i, j = 1, \dots, n_E\}.& \end{aligned}$$

The solution space is defined to contain the conditions of inextensibility and unshearability as follows:

$$V_S = \{(\mathbf{d}, \mathbf{w}) \in H^1(\mathcal{N}; \mathbb{R}^6) : \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i = 0, i = 1, \dots, n_E\}. \tag{2.8}$$

For a function $(\mathbf{d}, \mathbf{w}) \in V_S$ we consider the following norms on $H^1(\mathcal{N}; \mathbb{R}^3)$:

$$\|\mathbf{d}\|^2_{H^1(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{d}_i\|^2_{H^1(0,l_i;\mathbb{R}^3)}, \quad \|\mathbf{w}\|^2_{H^1(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_{H^1(0,l_i;\mathbb{R}^3)},$$

and the following norms on $L^2(\mathcal{N}; \mathbb{R}^3)$:

$$\|\mathbf{d}\|^2_{L^2(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{d}_i\|^2_{L^2(0,l_i;\mathbb{R}^3)}, \quad \|\mathbf{w}\|^2_{L^2(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_{L^2(0,l_i;\mathbb{R}^3)}.$$

The weak formulation for a single curved rod is obtained by first multiplying the first equation in (2.7) by a test function $\boldsymbol{\xi}$, the second equation in (2.7) by a test function $\boldsymbol{\zeta}$, and integrating over $[0, l]$ (we are dropping the subscript i in this calculation). The two equations added together give:

$$\rho_S A \int_0^l \partial_t^2 \mathbf{d} \cdot \boldsymbol{\xi} + \rho_S \int_0^l M \partial_t^2 \mathbf{w} \cdot \boldsymbol{\zeta} - \int_0^l \partial_s \mathbf{p} \cdot \boldsymbol{\xi} - \int_0^l \mathbf{f} \cdot \boldsymbol{\xi} \\ - \int_0^l \partial_s \mathbf{q} \cdot \boldsymbol{\zeta} + \int_0^l \mathbf{p} \cdot (\mathbf{t} \times \boldsymbol{\zeta}) = 0.$$

The terms that involve the partial derivative with respect to s can be rewritten by using integration by parts:

$$\rho_S A \int_0^l \partial_t^2 \mathbf{d} \cdot \boldsymbol{\xi} + \rho_S \int_0^l M \partial_t^2 \mathbf{w} \cdot \boldsymbol{\zeta} + \int_0^l \mathbf{p} \cdot \partial_s \boldsymbol{\xi} - \mathbf{p}(l) \cdot \boldsymbol{\xi}(l) + \mathbf{p}(0) \cdot \boldsymbol{\xi}(0) \\ - \int_0^l \mathbf{f} \cdot \boldsymbol{\xi} + \int_0^l \mathbf{q} \cdot \partial_s \boldsymbol{\zeta} - \mathbf{q}(l) \cdot \boldsymbol{\zeta}(l) + \mathbf{q}(0) \cdot \boldsymbol{\zeta}(0) + \int_0^l \mathbf{p} \cdot (\mathbf{t} \times \boldsymbol{\zeta}) = 0.$$

Finally, by using the constitutive relation and the condition of inextensibility and unshearability, we obtain the weak formulation for a single rod problem: find $(\mathbf{d}, \mathbf{w})$ such that

$$\rho_S A \int_0^l \partial_t^2 \mathbf{d} \cdot \boldsymbol{\xi} + \rho_S \int_0^l M \partial_t^2 \mathbf{w} \cdot \boldsymbol{\zeta} + \int_0^l Q H Q^T \partial_s \mathbf{w} \cdot \partial_s \boldsymbol{\zeta} \\ = \int_0^l \mathbf{f} \cdot \boldsymbol{\xi} + \mathbf{p}(l) \cdot \boldsymbol{\xi}(l) - \mathbf{p}(0) \cdot \boldsymbol{\xi}(0) + \mathbf{q}(l) \cdot \boldsymbol{\zeta}(l) - \mathbf{q}(0) \cdot \boldsymbol{\zeta}(0), \tag{2.9}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in H^1(0, l) \times H^1(0, l)$ that satisfy the condition of inextensibility and unshearability.

To get a weak formulation for the mesh net problem, we sum up the weak formulations for each local mesh component (i.e., curved rod, or strut). At each vertex, the boundary terms from (2.9) involving $\mathbf{p}$ and $\mathbf{q}$ will add up to zero due to the dynamic coupling conditions. The weak formulation for the mesh net problem then reads: find $(\mathbf{d}, \mathbf{w}) \in V_S$ such that

$$\rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t^2 \mathbf{d}_i \cdot \boldsymbol{\xi}_i + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t^2 \mathbf{w}_i \cdot \boldsymbol{\zeta}_i + \\ \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i = \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i \cdot \boldsymbol{\xi}_i, \tag{2.10}$$

for all test functions $(\boldsymbol{\xi}, \boldsymbol{\zeta}) = ((\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1), \ldots, (\boldsymbol{\xi}_{n_E}, \boldsymbol{\zeta}_{n_E})) \in V_S$.

2.1.4 Koiter shell and 1D mesh problem coupling

We will be assuming that the elastic mesh is always confined to the shell surface so that the following holds:

$$\bigcup_{i=1}^{n_E} \mathbf{P}_i([0, l_i]) \subset \Gamma = \boldsymbol{\varphi}(\overline{\omega}).$$

Since $\boldsymbol{\varphi}$ is injective on ω, functions $\boldsymbol{\pi}_i$, denoting the reparameterizations of the stent struts:

$$\boldsymbol{\pi}_i = \boldsymbol{\varphi}^{-1} \circ \mathbf{P}_i : [0, l_i] \to \overline{\omega}, \quad i = 1, \ldots, n_E,$$

are well defined. The reference configuration of the mesh defined on ω will be denoted by

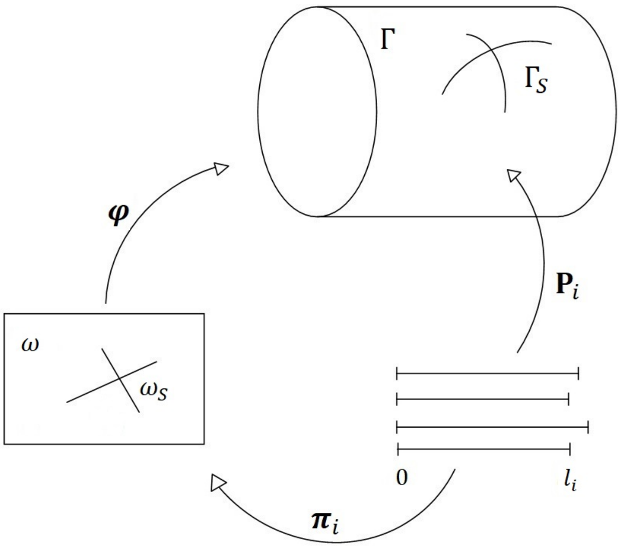

Figure 2.1: Parameterization of the mesh struts

$\omega_S = \bigcup_{i=1}^{n_E} \boldsymbol{\pi}_i([0, l_i])$, and of the mesh defined on Γ will be denoted by $\Gamma_S = \bigcup_{i=1}^{n_E} \mathbf{P}_i([0, l_i])$. See Fig. 2.1

The elastic mesh and the shell are coupled through the kinematic and dynamic coupling conditions. The kinematic coupling condition states that the displacement of the shell at the point $(R\cos\theta, R\sin\theta, z) \in \Gamma$, which is associated with the point $(z, \theta) \in \omega_S$ via the mapping $\boldsymbol{\varphi}$, is equal to the displacement of the stent at the point $s_i = \boldsymbol{\pi}_i^{-1}(z, \theta)$, that is associated to the same point $(z, \theta) \in \omega_S$ via the mapping $\boldsymbol{\pi}_i$. For a point $s_i \in [0, l_i]$ such that $\boldsymbol{\pi}_i(s_i) = (z, \theta) \in \omega_S$, the kinematic coupling condition reads:

$$\boldsymbol{\eta}(t, \boldsymbol{\pi}_i(s_i)) = \mathbf{d}_i(t, s_i). \tag{2.11}$$

The dynamic coupling condition describes the balance of forces. The force exerted by the Koiter shell onto the mesh is balanced by the force exerted by the mesh onto the Koiter shell. More precisely, let $J_i = \boldsymbol{\pi}_i([0, l_i])$, and

$$\langle \delta_{J_i}, f \rangle = \int_{J_i} f d\gamma_i, \quad i = 1, \ldots, n_E,$$

where $d\gamma_i$ is the curve element associated with the parameterization $\boldsymbol{\pi}_i$. The weak formulation of the shell (2.5) can then be written as:

$$
\begin{aligned}
\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle &= \sum_{i=1}^{n_E} \langle \delta_{J_i}, \mathbf{f}\cdot\boldsymbol{\psi} R\rangle \\
&= \sum_{i=1}^{n_E} \int_{J_i} \mathbf{f}(z,\theta)\cdot\boldsymbol{\psi}(z,\theta) R d\gamma_i \\
&= \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}(\boldsymbol{\pi}_i(s))\cdot\boldsymbol{\psi}(\boldsymbol{\pi}_i(s))\|\boldsymbol{\pi}_i'(s)\| R ds.
\end{aligned}
$$

If we denote by $\mathbf{f}_i$ the force exerted by the i-th mesh strut onto the shell, by force balance, the right-hand side has to be equal to $-\sum_{i=1}^{n_E}\int_0^{l_i} \mathbf{f}_i(s)\cdot\boldsymbol{\xi}_i(s)ds$. Thus, $\mathbf{f}(\boldsymbol{\pi}_i(s_i))\|\boldsymbol{\pi}_i'(s_i)\|R = -\mathbf{f}_i(s_i)$, i.e. $\mathbf{f}(\boldsymbol{\pi}_i(s_i))R = -\dfrac{\mathbf{f}_i(s_i)}{\|\boldsymbol{\pi}_i'(s_i)\|}$, $s_i \in (0,l_i)$. For a point $(z,\theta) = \boldsymbol{\pi}_i(s_i) \in \omega_S$, which came from an $s_i \in (0,l_i)$, the dynamic coupling condition reads: $\mathbf{f}R = -\dfrac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|}$. For an arbitrary point $(z,\theta)\in\omega$, the dynamic coupling condition reads:

$$
\mathbf{f}R = -\sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|}\delta_{J_i}. \tag{2.12}
$$

Now, the weak formulation for the coupled mesh-shell problem reads:

$$
\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta}\cdot\boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle = -\sum_{i=1}^{n_E}\langle \delta_{J_i}, \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|}\boldsymbol{\psi}\rangle, \tag{2.13}
$$

for all test functions $\boldsymbol{\psi} \in V_K$. Here $\mathbf{f}_i$'s are defined in (2.10), where the test functions $\boldsymbol{\xi}_i$ are such that $\boldsymbol{\psi}\circ\boldsymbol{\pi}_i = \boldsymbol{\xi}_i$. The coupled mesh-shell weak solution space is given by:

$$
V_{KS} = \{(\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_K \times V_S : \boldsymbol{\eta}\circ\boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E}(0,l_i)\},
$$

where we denoted $\boldsymbol{\eta}\circ\boldsymbol{\pi} = (\boldsymbol{\eta}\circ\boldsymbol{\pi}_1, \ldots, \boldsymbol{\eta}\circ\boldsymbol{\pi}_{n_E})$.

The *dynamic* coupled mesh-shell problem presented here is an extension of the coupled mesh-shell problem first studied in [18] for the static case.

2.1.5 The fluid-composite structure coupling

From now on, by 'structure' we will refer to the Koiter shell coupled with the 1D elastic mesh described above. The coupling between the fluid and the structure is defined by two sets of coupling conditions: the kinematic and dynamic coupling conditions, satisfied at the fixed, lateral boundary Γ, giving rise to a linear fluid-structure coupling. The coupling conditions impose continuity of velocity and balance of contact forces. Let us emphasize that in our work, the dynamic coupling condition also reflects the presence of a 1D elastic mesh at the fluid-structure interface. The coupling conditions read:

- the kinematic condition: $\partial_t \boldsymbol{\eta} = \mathbf{u}|_\Gamma \circ \boldsymbol{\varphi}$ on $(0,T) \times \omega$,
- the dynamic condition:

$$\rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} = -J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}), \text{ on } (0,T) \times \omega,$$

where J denotes the Jacobian of the transformation from cylindrical to Cartesian coordinates, and $\mathbf{n}$ denotes the outer unit normal on Γ. For the linear fluid-structure interaction problem considered here, the Jacobian J is equal to R.

In summary, we study the following fluid-structure interaction problem: **Problem 1.** Find $(\mathbf{u}, p, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ such that

$$\left.\begin{array}{rcl} \rho_F \partial_t \mathbf{u} &=& \nabla \cdot \boldsymbol{\sigma} \\ \nabla \cdot \mathbf{u} &=& 0 \end{array}\right\} \text{ in } (0,T) \times \Omega, \tag{2.14}$$

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{\eta} &=& \mathbf{u} \circ \boldsymbol{\varphi} \\ \rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{||\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}||} \delta_{J_i} &=& -J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}) \end{array}\right\} \text{ on } (0,T) \times \omega, \tag{2.15}$$

$$\left.\begin{array}{rcl} \rho_S A_i \partial_t^2 \mathbf{d}_i &=& \partial_s \mathbf{p}_i + \mathbf{f}_i \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &=& \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i \\ 0 &=& \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i \\ 0 &=& \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i \end{array}\right\} \text{ on } (0,T) \times (0,l_i). \tag{2.16}$$

Problem (2.14)-(2.16) is supplemented with the following set of boundary and the initial conditions:

$$\left\{\begin{array}{rcll} p &=& P_{in/out}(t), & \text{on } (0,T) \times \Gamma_{in/out}, \\ \mathbf{u} \times \mathbf{e}_z &=& 0, & \text{on } (0,T) \times \Gamma_{in/out}, \\ \boldsymbol{\eta}(t,0,\theta) &=& \boldsymbol{\eta}(t,L,\theta) = 0, & \text{on } (0,T) \times (0,2\pi), \\ \partial_z \eta_r(t,0,\theta) &=& \partial_z \eta_r(t,L,\theta) = 0, & \text{on } (0,T) \times (0,2\pi), \\ \boldsymbol{\eta}(t,z,0) &=& \boldsymbol{\eta}(t,z,2\pi), & \text{on } (0,T) \times (0,L), \\ \partial_\theta \eta_r(t,z,0) &=& \partial_\theta \eta_r(t,z,2\pi), & \text{on } (0,T) \times (0,L), \end{array}\right. \tag{2.17}$$

$$\begin{array}{c} \mathbf{u}(0) = \mathbf{u}_0, \ \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0, \ \partial_t \boldsymbol{\eta}(0) = \mathbf{v}_0, \\ \mathbf{d}_i(0) = \mathbf{d}_{0i}, \ \partial_t \mathbf{d}_i(0) = \mathbf{k}_{0i}, \ \mathbf{w}_i(0) = \mathbf{w}_{0i}, \ \partial_t \mathbf{w}_i(0) = \mathbf{z}_{0i}. \end{array} \tag{2.18}$$

2.2 The energy of the coupled fluid-mesh-shell problem

We formally prove that problem (2.14)-(2.16) satisfies the following energy inequality:

$$\frac{d}{dt}E(t) + D(t) \leq C(P_{in}(t), P_{out}(t)), \tag{2.19}$$

where $E(t)$ denotes the total energy of the coupled problem (the sum of the kinetic and elastic energy), $D(t)$ denotes dissipation due to fluid viscosity, and $C(P_{in}(t), P_{out}(t))$ is a constant which depends only on the L^2-norms of the inlet and outlet pressure data. More precisely, if we denote by $\|\mathbf{w}\|_m$ and $\|\mathbf{w}\|_r$ the following norms associated with the elastic energy of the elastic mesh:

$$\|\mathbf{w}\|_m^2 := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|_m^2 = \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{w}_i \cdot \mathbf{w}_i,$$

$$\|\mathbf{w}\|_r^2 := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|_r^2 = \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \mathbf{w}_i,$$

and by $\|\boldsymbol{\eta}\|_{L^2(R;\omega)}$ the weighted L^2 norm on ω, with the weight R associated with the geometry of the Koiter shell (Jacobian):

$$\|\boldsymbol{\eta}\|_{L^2(R;\omega)}^2 := \int_\omega |\boldsymbol{\eta}|^2 R \, d\omega,$$

then the total energy of the coupled FSI problem is defined by

$$\begin{aligned} E(t) = {} & \frac{\rho_F}{2}\|\mathbf{u}\|_{L^2(\Omega)}^2 + \frac{\rho_K h}{2}\|\partial_t \boldsymbol{\eta}\|_{L^2(R;\omega)}^2 + \frac{\rho_S}{2}\sum_{i=1}^{n_E} A_i \|\partial_t \mathbf{d}_i\|_{L^2(0,l_i)}^2 \\ & + \frac{\rho_S}{2}\|\partial_t \mathbf{w}\|_m^2 + \frac{1}{2}\langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\eta}\rangle + \|\mathbf{w}\|_r^2, \end{aligned}$$

while $D(t)$ is given by

$$D(t) = 2\mu_F \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}^2.$$

Notice that the norm $\|\cdot\|_m$ is equivalent to the standard $L^2(\mathcal{N})$ norm.

Proposition 2.1 There exists a constant $C > 0$ such that

$$\|\mathbf{w}\|_r^2 \leq C\|\mathbf{w}\|_{H^1(\mathcal{N})}^2.$$

Proof. Since Q_i is orthogonal and H_i positive definite, for each $i = 1, \ldots, n_E$, the following inequality holds:

$$\begin{aligned}
\|\mathbf{w}_i\|_r^2 &= \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \mathbf{w}_i = \int_0^{l_i} H_i Q_i^T \partial_s \mathbf{w}_i \cdot Q_i^T \partial_s \mathbf{w}_i \\
&\leq \int_0^{l_i} \lambda_{max}(H_i) Q_i^T \partial_s \mathbf{w}_i \cdot Q_i^T \partial_s \mathbf{w}_i = \lambda_{max}(H_i) \|Q_i^T \partial_s \mathbf{w}_i\|_{L^2(0,l_i)}^2 \\
&= \lambda_{max}(H_i) \|\partial_s \mathbf{w}_i\|_{L^2(0,l_i)}^2 \leq \lambda_{max}(H_i) \left(\|\partial_s \mathbf{w}_i\|_{L^2(0,l_i)}^2 + \|\mathbf{w}_i\|_{L^2(0,l_i)}^2 \right) \\
&\leq \lambda_{max}(H_i) \|\mathbf{w}_i\|_{H^1(0,l_i)}^2,
\end{aligned}$$

where $\lambda_{max}(H_i)$ is the maximum eigenvalue of matrix H_i. □

To derive the energy inequality (2.19) we first multiply the first equation in (2.1) by $\mathbf{u}$ and integrate by parts over Ω to obtain:

$$\begin{aligned}
&\int_\Omega \rho_F \partial_t \mathbf{u} \cdot \mathbf{u} - \int_\Omega \nabla \cdot (-pI + 2\mu_F \mathbf{D}(\mathbf{u})) \cdot \mathbf{u} \\
&= \frac{\rho_F}{2} \frac{d}{dt} \int_\Omega |\mathbf{u}|^2 - \int_\Omega \nabla \cdot ((-pI + 2\mu_F \mathbf{D}(\mathbf{u}))\mathbf{u}) + \int_\Omega (-pI + 2\mu_F \mathbf{D}(\mathbf{u})) : \nabla \mathbf{u} \\
&= \frac{\rho_F}{2} \frac{d}{dt} \int_\Omega |\mathbf{u}|^2 - \int_{\partial\Omega} (-pI + 2\mu_F \mathbf{D}(\mathbf{u}))\mathbf{n} \cdot \mathbf{u} - \int_\Omega pI : \nabla \mathbf{u} + 2\mu_F \int_\Omega \mathbf{D}(\mathbf{u}) : \nabla \mathbf{u} \\
&= \frac{\rho_F}{2} \frac{d}{dt} \int_\Omega |\mathbf{u}|^2 - \int_{\partial\Omega} (-pI + 2\mu_F \mathbf{D}(\mathbf{u}))\mathbf{n} \cdot \mathbf{u} - \int_\Omega p \nabla \cdot \mathbf{u} + 2\mu_F \int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) \\
&= \frac{\rho_F}{2} \frac{d}{dt} \int_\Omega |\mathbf{u}|^2 - \int_{\partial\Omega} (-pI + 2\mu_F \mathbf{D}(\mathbf{u}))\mathbf{n} \cdot \mathbf{u} + 2\mu_F \int_\Omega |\mathbf{D}(\mathbf{u})|^2,
\end{aligned}$$

where we have used $\nabla \mathbf{u} : \nabla \mathbf{u} = \mathrm{tr}(\nabla \mathbf{u} \nabla^T \mathbf{u}) = \mathrm{tr}(\nabla^T \mathbf{u} \nabla \mathbf{u}) = \nabla^T \mathbf{u} : \nabla^T \mathbf{u}$.
To calculate the boundary integral over $\partial\Omega$, we first notice that on $\Gamma_{in/out}$ the boundary condition $\mathbf{u} \times \mathbf{e}_z = 0$ gives $u_x = u_y = 0$. Using divergence-free condition, we also obtain $\partial_z u_z = 0$. Now, since the normal to $\Gamma_{in/out}$ is equal to $\mathbf{n} = (\mp 1, 0, 0)$, we get:

$$\begin{aligned}
&- \int_{\partial\Omega} (-pI + 2\mu_F \mathbf{D}(\mathbf{u}))\mathbf{n} \cdot \mathbf{u} \\
&= - \int_\Gamma \boldsymbol{\sigma} \mathbf{n} \cdot \mathbf{u} - \int_{\Gamma_{in} \cup \Gamma_{out}} (-pI + 2\mu_F \mathbf{D}(\mathbf{u}))\mathbf{n} \cdot \mathbf{u} \\
&= - \int_\Gamma \boldsymbol{\sigma} \mathbf{n} \cdot \mathbf{u} - \int_{\Gamma_{in}} p u_z + \int_{\Gamma_{out}} p u_z.
\end{aligned} \tag{2.20}$$

To calculate the boundary integral over Γ we first multiply the Koiter shell equation (2.6) by $\partial_t \boldsymbol{\eta}$ and integrate over ω:

$$\begin{aligned}
\int_\omega \mathbf{f} \cdot \partial_t \boldsymbol{\eta} R &= \rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\eta} R + \langle \boldsymbol{\mathcal{L}} \boldsymbol{\eta}, \partial_t \boldsymbol{\eta} \rangle \\
&= \frac{\rho_K h}{2} \frac{d}{dt} \int_\omega \partial_t \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\eta} R + \frac{1}{2} \frac{d}{dt} \langle \boldsymbol{\mathcal{L}} \boldsymbol{\eta}, \boldsymbol{\eta} \rangle \\
&= \frac{\rho_K h}{2} \frac{d}{dt} \|\partial_t \boldsymbol{\eta}\|_{L^2(R;\omega)}^2 + \frac{1}{2} \frac{d}{dt} \langle \boldsymbol{\mathcal{L}} \boldsymbol{\eta}, \boldsymbol{\eta} \rangle.
\end{aligned}$$

Next, we want to use $(\partial_t \mathbf{d}_i, \partial_t \mathbf{w}_i), i = 1, \ldots, n_E$, as test functions in the weak formulation for the mesh problem (2.10). Before doing so, we need to check if these test functions are *admissible*, i.e. if they satisfy the condition of inextensibility and unshearability $\partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i = 0, i = 1, \ldots, n_E$. We differentiate this condition with respect to t and use the fact that $\mathbf{t}_i$ do not depend on t, to obtain

$$\partial_s(\partial_t \mathbf{d}_i) + \mathbf{t}_i \times (\partial_t \mathbf{w}_i) = 0, \quad i = 1, \ldots, n_E,$$

which implies that, indeed, $(\partial_t \mathbf{d}_i, \partial_t \mathbf{w}_i) \in V_S,\ i = 1, \ldots, n_E$.

By using $(\partial_t \mathbf{d}, \partial_t \mathbf{w}) = ((\partial_t \mathbf{d}_1, \partial_t \mathbf{w}_1), \ldots, (\partial_t \mathbf{d}_{n_E}, \partial_t \mathbf{w}_{n_E}))$ as a test function in the weak formulation for the mesh problem, we obtain:

$$\begin{aligned}\sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i \cdot \partial_t \mathbf{d}_i &= \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t^2 \mathbf{d}_i \cdot \partial_t \mathbf{d}_i + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t^2 \mathbf{w}_i \cdot \partial_t \mathbf{w}_i \\ &\quad + \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \partial_t \mathbf{w}_i \\ &= \frac{\rho_S}{2} \frac{d}{dt} \sum_{i=1}^{n_E} A_i \|\partial_t \mathbf{d}_i\|^2_{L^2(0,l_i)} + \frac{\rho_S}{2} \frac{d}{dt} \sum_{i=1}^{n_E} \|\partial_t \mathbf{w}_i\|^2_m \\ &\quad + \frac{d}{dt} \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_r.\end{aligned}$$

By enforcing the kinematic and dynamic boundary conditions on Γ, and recalling that the Jacobian of the transformation between Γ and ω is $J = R$, we obtain:

$$\begin{aligned}-\int_\Gamma \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{u} = -\int_\omega J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}) \cdot \partial_t \boldsymbol{\eta} = \int_\omega \mathbf{f} \cdot \partial_t \boldsymbol{\eta} R + \sum_{i=1}^{n_E} \int_{J_i} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} \cdot \partial_t \boldsymbol{\eta} \\ = \int_\omega \mathbf{f} \cdot \partial_t \boldsymbol{\eta} R + \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i \cdot \partial_t \boldsymbol{\eta} \circ \boldsymbol{\pi}_i = \int_\omega \mathbf{f} \cdot \partial_t \boldsymbol{\eta} R + \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i \cdot \partial_t \mathbf{d}_i.\end{aligned} \tag{2.21}$$

By inserting the expressions for $\mathbf{f}$ and $\mathbf{f}_i$ from the shell and mesh problems into (2.21), we get:

$$\begin{aligned}-\int_\Gamma \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{u} &= \frac{\rho_K h}{2} \frac{d}{dt} \|\partial_t \boldsymbol{\eta}\|^2_{L^2(R;\omega)} + \frac{1}{2} \frac{d}{dt} \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\eta} \rangle + \frac{\rho_S}{2} \frac{d}{dt} \sum_{i=1}^{n_E} A_i \|\partial_t \mathbf{d}_i\|^2_{L^2(0,l_i)} \\ &\quad + \frac{\rho_S}{2} \frac{d}{dt} \sum_{i=1}^{n_E} \|\partial_t \mathbf{w}_i\|^2_m + \frac{d}{dt} \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_r.\end{aligned} \tag{2.22}$$

Finally, by replacing the trace of the normal stress on Γ in (2.20) by (2.22), one obtains the following energy equality:

$$\begin{aligned}
&\frac{\rho_F}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega)} + 2\mu_F\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)} + \frac{\rho_K h}{2}\frac{d}{dt}\|\partial_t\boldsymbol{\eta}\|^2_{L^2(R;\omega)} \\
&\quad + \frac{1}{2}\frac{d}{dt}\langle\boldsymbol{\mathcal{L}}\boldsymbol{\eta},\boldsymbol{\eta}\rangle + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E} A_i\|\partial_t\mathbf{d}_i\|^2_{L^2(0,l_i)} + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}\|\partial_t\mathbf{w}_i\|^2_m \\
&\quad + \frac{d}{dt}\sum_{i=1}^{n_E}\|\mathbf{w}_i\|^2_r = \int_{\Gamma_{in}} pu_z - \int_{\Gamma_{out}} pu_z.
\end{aligned} \tag{2.23}$$

The right-hand side is equal to

$$\int_{\Gamma_{in}} P_{in}(t)u_z - \int_{\Gamma_{out}} P_{out}(t)u_z.$$

Using the trace theorem, Korn inequality and Cauchy inequality (with ε), one can estimate:

$$\begin{aligned}
\left|\int_{\Gamma_{in/out}} P_{in/out}(t)u_z\right| &\le C|P_{in/out}|\|\mathbf{u}\|_{H^1(\Omega)} \le C|P_{in/out}|\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)} \\
&\le \frac{C}{2\varepsilon}|P_{in/out}|^2 + \frac{C\varepsilon}{2}\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}.
\end{aligned}$$

We note that, indeed, the fluid velocity $\mathbf{u}$ satisfies the conditions for Korn inequality (Theorem 6.3-4 in [29]). Namely, the boundary condition $\mathbf{u}\times\mathbf{e}_z = 0$ on $\Gamma_{in/out}$ gives us $u_x = u_y = 0$ and $\partial_z u_z = 0$. Using the kinematic coupling condition $u_z = \partial_t\eta_z$ (on ω), we obtain that $u_z = 0$, so $\mathbf{u} = 0$ on $\Gamma_{in/out}$. Finally, by choosing ε such that $\frac{C\varepsilon}{2} \le \mu_F$, we get the energy inequality (2.19).

2.3 The operator splitting scheme

Our goal is to approximate the coupled FSI problem using time-discretization via operator splitting, and then prove that solutions to the approximate problems converge to a weak solution of the continuous problem, as the time-discretization step tends to zero.

We use the Lie splitting scheme which can be summarized as follows. Let $N \in \mathbb{N}, \Delta t = T/N$ and $t_n = n\Delta t$. Consider the following initial-value problem

$$\begin{aligned}
\frac{d\phi}{dt} + A\phi = 0 \quad \text{in} \quad (0,T), \\
\phi(0) = \phi_0,
\end{aligned}$$

where A is an operator defined on a Hilbert space, and A can be written as $A = A_1 + A_2$. Set $\phi^0 = \phi_0$ and for $n = 0,\dots,N-1, i = 1,2$, compute $\phi^{n+\frac{i}{2}}$ by solving

$$\frac{d\phi_i}{dt} + A_i\phi_i = 0 \quad \text{in} \quad (t_n, t_{n+1}),$$

$$\phi_i(t_n) = \phi^{n+\frac{i-1}{2}},$$

and then set $\phi^{n+\frac{i}{2}} = \phi_i(t_{n+1})$.

To perform the time-discretization via operator splitting, we need to rewrite our FSI problem as a first order system in time. This will be done by replacing the second-order time derivative of $\boldsymbol{\eta}$, with the first-order time derivative of the Koiter shell velocity $\mathbf{v} = \partial_t \boldsymbol{\eta}$, by replacing the second-order time derivative of $\mathbf{d}$ by the first-order time derivative of the mesh velocity $\mathbf{k} = \partial_t \mathbf{d}$, and by replacing the second-order time derivative of $\mathbf{w}$ by the first-order time derivative of the rotation velocity $\mathbf{z} = \partial_t \mathbf{w}$.

We apply this approach to split Problem 1 into a fluid and a structure subproblem. There are many different ways to split the coupled problem into a fluid and a structure subproblem. Only certain splitting strategies would lead to a stable and convergent scheme. Here, we define a structure and a fluid subproblem for the Lie splitting scheme that will, indeed, provide a convergent scheme, converging to a weak solution of the coupled, continuous problem.

2.3.1 The structure subproblem

In this step we solve an elastodynamics problem for the location of the deformable boundary, which is defined by the dynamic coupling condition involving only the elastic energy of the structure. The motion of the structure is driven by the initial velocity of the structure, obtained, using the kinematic coupling condition, from the trace of the fluid velocity calculated in the previous time step. The fluid velocity $\mathbf{u}$ remains unchanged in this step. The structure subproblem reads: for a given $(\mathbf{u}^n, \boldsymbol{\eta}^n, \mathbf{v}^n, \mathbf{d}^n, \mathbf{w}^n, \mathbf{k}^n, \mathbf{z}^n)$, calculated in the previous time step, find $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{v}, \mathbf{d}, \mathbf{w}, \mathbf{k}, \mathbf{z})$ such that

$$\partial_t \mathbf{u} = 0, \quad \text{in} \quad (t_n, t_{n+1}) \times \Omega,$$

$$\begin{cases} \rho_K h \partial_t \mathbf{v} R + \mathcal{L}\boldsymbol{\eta} + \rho_S \displaystyle\sum_{i=1}^{n_E} A_i \frac{\partial_t \mathbf{k}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} + \rho_S \displaystyle\sum_{i=1}^{n_E} M_i \frac{\partial_t \mathbf{z}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} \\ + \displaystyle\sum_{i=1}^{n_E} Q_i H_i Q_i^T \frac{\partial_s \mathbf{w}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} = 0, \quad \text{in} \quad (t_n, t_{n+1}) \times \omega, \\ \boldsymbol{\eta} \circ \boldsymbol{\pi}_i = \mathbf{d}_i, \quad \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \end{cases}$$

$$\begin{cases} \partial_t \boldsymbol{\eta} = \mathbf{v}, \quad \text{in} \quad (t_n, t_{n+1}) \times \omega, \\ \partial_t \mathbf{d}_i = \mathbf{k}_i, \quad \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots n_E, \\ \partial_t \mathbf{w}_i = \mathbf{z}_i, \quad \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \\ \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i = 0, \quad \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \end{cases}$$

with boundary data:

$$\begin{cases} \boldsymbol{\eta}(t,0,\theta) = \boldsymbol{\eta}(t,L,\theta) = \partial_z \eta_r(t,0,\theta) = \partial_z \eta_r(t,L,\theta) = 0, & \text{on} \quad (t_n,t_{n+1}) \times (0,2\pi), \\ \boldsymbol{\eta}(t,z,0) = \boldsymbol{\eta}(t,z,2\pi), \ \ \partial_\theta \eta_r(t,z,0) = \partial_\theta \eta_r(t,z,2\pi), & \text{on} \quad (t_n,t_{n+1}) \times (0,L), \end{cases}$$

and initial data:

$$\begin{cases} \mathbf{u}(t_n) = \mathbf{u}^n, \boldsymbol{\eta}(t_n) = \boldsymbol{\eta}^n, \mathbf{v}(t_n) = \mathbf{v}^n, \mathbf{d}(t_n) = \mathbf{d}^n, \\ \mathbf{w}(t_n) = \mathbf{w}^n, \mathbf{k}(t_n) = \mathbf{k}^n, \mathbf{z}(t_n) = \mathbf{z}^n. \end{cases}$$

Then set $\mathbf{u}^{n+1/2} = \mathbf{u}(t_{n+1}), \boldsymbol{\eta}^{n+1/2} = \boldsymbol{\eta}(t_{n+1}), \mathbf{v}^{n+1/2} = \mathbf{v}(t_{n+1}), \mathbf{d}^{n+1/2} = \mathbf{d}(t_{n+1}), \mathbf{w}^{n+1/2} = \mathbf{w}(t_{n+1}), \mathbf{k}^{n+1/2} = \mathbf{k}(t_{n+1}), \mathbf{z}^{n+1/2} = \mathbf{z}(t_{n+1})$.

2.3.2 The fluid subproblem

In this step we solve the Stokes equations for the fluid, with a Robin-type boundary condition on Γ, which is obtained by using the remaining part of the dynamic coupling condition, not used in the structure subproblem. Thus, the boundary condition involves the first-order time derivative term corresponding to the shell inertia, and the trace of the fluid normal stress on Γ. Since the fluid and the elastic mesh "feel" each other only through the motion of the shell, meaning that the fluid motion affects the shell motion, and then the shell motion affects the mesh motion whose elastodynamics is influenced by the presence of the mesh, we *exclude* the mesh from the fluid subproblem. Namely, since we are working with weak solutions of Leray type, the fluid velocity has no trace on the mesh domain since it is a one-dimensional set. Thus, in this step, the structure displacement, the velocity of the mesh displacement, and the velocity of infinitesimal rotation of cross sections remain unchanged. The fluid subproblem reads: for a given $(\mathbf{u}^{n+\frac{1}{2}}, \boldsymbol{\eta}^{n+\frac{1}{2}}, \mathbf{v}^{n+\frac{1}{2}}, \mathbf{d}^{n+\frac{1}{2}}, \mathbf{w}^{n+\frac{1}{2}}, \mathbf{k}^{n+\frac{1}{2}}, \mathbf{z}^{n+\frac{1}{2}})$, find $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{v}, \mathbf{d}, \mathbf{w}, \mathbf{k}, \mathbf{z})$ such that:

$$\begin{cases} \rho_F \partial_t \mathbf{u} = \nabla \cdot \boldsymbol{\sigma}, & \text{in} \quad (t_n,t_{n+1}) \times \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in} \quad (t_n,t_{n+1}) \times \Omega, \end{cases}$$
$$\begin{cases} \rho_K h \partial_t \mathbf{v} R = -J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}), & \text{in} \quad (t_n,t_{n+1}) \times \omega, \\ \mathbf{u}|_\Gamma \circ \boldsymbol{\varphi} = \mathbf{v}, & \text{in} \quad (t_n,t_{n+1}) \times \omega, \end{cases}$$
$$\begin{cases} p = P_{in/out}(t), & \text{on} \quad (t_n,t_{n+1}) \times \Gamma_{in/out}, \\ \mathbf{u} \times \mathbf{e}_z = 0, & \text{on} \quad (t_n,t_{n+1}) \times \Gamma_{in/out}, \end{cases}$$

$$\begin{cases} \partial_t \boldsymbol{\eta} = 0, & \text{in} \quad (t_n, t_{n+1}) \times \omega, \\ \partial_t \mathbf{d}_i = 0, & \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \\ \partial_t \mathbf{w}_i = 0, & \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \end{cases}$$

$$\begin{cases} \partial_t \mathbf{k}_i = 0, & \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \\ \partial_t \mathbf{z}_i = 0, & \text{in} \quad (t_n, t_{n+1}) \times (0, l_i), \quad i = 1, \dots, n_E, \end{cases}$$

with initial data:

$$\begin{cases} \mathbf{u}(t_n) = \mathbf{u}^{n+1/2}, \boldsymbol{\eta}(t_n) = \boldsymbol{\eta}^{n+1/2}, \mathbf{v}(t_n) = \mathbf{v}^{n+1/2}, \mathbf{d}(t_n) = \mathbf{d}^{n+1/2}, \\ \mathbf{w}(t_n) = \mathbf{w}^{n+1/2}, \mathbf{k}(t_n) = \mathbf{k}^{n+1/2}, \mathbf{z}(t_n) = \mathbf{z}^{n+1/2}. \end{cases}$$

Then set $\mathbf{u}^{n+1} = \mathbf{u}(t_{n+1}), \boldsymbol{\eta}^{n+1} = \boldsymbol{\eta}(t_{n+1}), \mathbf{v}^{n+1} = \mathbf{v}(t_{n+1}), \mathbf{d}^{n+1} = \mathbf{d}(t_{n+1})$, $\mathbf{w}^{n+1} = \mathbf{w}(t_{n+1}), \mathbf{k}^{n+1} = \mathbf{k}(t_{n+1}), \mathbf{z}^{n+1} = \mathbf{z}(t_{n+1})$.

Crucial for a design of a stable scheme is the inclusion of structure inertia into the fluid subproblem, which guarantees energy balance at the time-discrete level, thereby avoiding stability problems due to the so called *added mass effect*. Added mass effect is used to describe the elastodynamics of structures interacting with fluids with comparable densities, for which there is a significant exchange of energy between the fluid and structure motion, potentially causing instabilities is schemes that do not approximate well the energy exchange that occurs at the continuous level.

2.4 Existence of weak solutions

2.4.1 Function spaces

Motivated by the energy inequality (2.19), we define the following evolution spaces associated with the fluid problem, the Koiter shell problem, the mesh problem and the coupled mesh-shell problem:

- $V_F(0,T) = L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V_F)$,
  where V_F is defined by (2.2),
- $V_K(0,T) = W^{1,\infty}(0,T; L^2(R;\omega)) \cap L^\infty(0,T; V_K)$,
  where V_K is defined by (2.4),
- $V_S(0,T) = W^{1,\infty}(0,T; L^2(\mathcal{N})) \cap L^\infty(0,T; V_S)$,
  where V_S is defined by (2.8),
- $V_{KS}(0,T) = \{(\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_K(0,T) \times V_S(0,T) : \boldsymbol{\eta} \circ \boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E} (0, l_i)\}$.

The solution space for the coupled fluid-mesh-shell interaction problem involves the kinematic coupling condition, which is, thus, enforced in a strong way:

$$V(0,T) = \{(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_F(0,T) \times V_{KS}(0,T) : \mathbf{u} \circ \boldsymbol{\varphi} = \partial_t \boldsymbol{\eta} \text{ on } \omega\}.$$

The associated test space is given by:

$$Q(0,T) = \{(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in C_c^1([0,T); V_F \times V_{KS}) : \boldsymbol{\upsilon} \circ \boldsymbol{\varphi} = \boldsymbol{\psi} \text{ on } \omega\}.$$

2.4.2 Definition of a weak solution

We are now in a position to state a definition of weak solutions of our fluid-mesh-shell interaction problem, with the fluid flow in Ω.

Definition 2.2 We say that $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V(0,T)$ is a weak solution of Problem 1 if for all test functions $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in Q(0,T)$ the following equality holds:

$$\begin{aligned}
&- \rho_F \int_0^T \int_\Omega \mathbf{u} \cdot \partial_t \boldsymbol{\upsilon} + 2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{\upsilon}) - \rho_K h \int_0^T \int_\omega \partial_t \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\psi} R \\
&+ \int_0^T a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) - \rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} \partial_t \mathbf{d}_i \cdot \partial_t \boldsymbol{\xi}_i - \rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i \partial_t \mathbf{w}_i \cdot \partial_t \boldsymbol{\zeta}_i \\
&+ \int_0^T a_S(\mathbf{w}, \boldsymbol{\zeta}) = \int_0^T \langle F(t), \boldsymbol{\upsilon} \rangle_{\Gamma_{in/out}} + \rho_F \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\upsilon}(0) + \rho_K h \int_\omega \mathbf{v}_0 \cdot \boldsymbol{\psi}(0) R \\
&+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{k}_{0i} \cdot \boldsymbol{\xi}_i(0) + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{z}_{0i} \cdot \boldsymbol{\zeta}_i(0),
\end{aligned}$$

where

$$a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) = \langle \boldsymbol{\mathcal{L}} \boldsymbol{\eta}, \boldsymbol{\psi} \rangle,$$

$$a_S(\mathbf{w}, \boldsymbol{\zeta}) = \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i,$$

and

$$\langle F(t), \boldsymbol{\upsilon} \rangle_{\Gamma_{in/out}} = P_{in}(t) \int_{\Gamma_{in}} \upsilon_z - P_{out}(t) \int_{\Gamma_{out}} \upsilon_z.$$

2.4.3 Statement of Main Existence Result

Our goal is to prove the existence of such weak solutions. More precisely, we prove the following main result of this chapter:

Theorem 2.3 Let $\mathbf{u}_0 \in L^2(\Omega)$, $\boldsymbol{\eta}_0 \in H^1(\omega)$, $\mathbf{v}_0 \in L^2(R; \omega)$, $(\mathbf{d}_0, \mathbf{w}_0) \in V_S$, $(\mathbf{k}_0, \mathbf{z}_0) \in L^2(\mathcal{N}; \mathbb{R}^6)$ be such that

$$\nabla \cdot \mathbf{u}_0 = 0, \ (\mathbf{u}_0|_\Gamma \circ \boldsymbol{\varphi}) \cdot \mathbf{e}_r = (\mathbf{v}_0)_r, \ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0, \ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0.$$

Furthermore, let all the physical constants be positive: $\rho_K, \rho_S, \rho_F, \lambda, \mu, \mu_F > 0$ and $A_i > 0, \forall i = 1, \dots, n_E$, and let $P_{in/out} \in L^2_{loc}(0,\infty)$. Then for every $T > 0$ there exists a weak solution to Problem 1 in the sense of Definition 2.2.

The rest of this chapter is dedicated to designing a constructive proof of this existence result.

2.5 Approximate solutions

We construct approximate solutions to Problem 1 by semi-discretizing the subproblems defined in Sec. 2.3 using the Backward Euler scheme. Let $\Delta t = T/N$ be the time-discretization parameter so that the time interval $(0,T)$ is subdivided into N subintervals of width Δt. We define the vector of unknown approximate solutions

$$\mathbf{X}_N^{n+i/2} = (\mathbf{u}_N^{n+i/2}, \boldsymbol{\eta}_N^{n+i/2}, \mathbf{v}_N^{n+i/2}, \mathbf{d}_N^{n+i/2}, \mathbf{w}_N^{n+i/2}, \mathbf{k}_N^{n+i/2}, \mathbf{z}_N^{n+i/2}),$$

$n = 0, 1, \dots, N-1, i = 1, 2$, where $i = 1, 2$ denotes the solution of the structure and the fluid subproblem, respectively. We semi-discretize the problem so that the discrete version of the energy inequality (2.19) is preserved at every time step. We define the semi-discrete versions of the kinetic and elastic energy, and of dissipation, by the following:

$$\begin{aligned} E_N^{n+i/2} = \rho_F \int_\Omega |\mathbf{u}^{n+i/2}|^2 + \rho_K h \int_\omega |\mathbf{v}^{n+i/2}|^2 R + a_K(\boldsymbol{\eta}^{n+i/2}, \boldsymbol{\eta}^{n+i/2}) \\ + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} |\mathbf{k}_i^{n+i/2}|^2 + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i |\mathbf{z}_i^{n+i/2}|^2 \\ + a_S(\mathbf{w}^{n+i/2}, \mathbf{w}^{n+i/2}), \end{aligned} \tag{2.24}$$

$$D_N^{n+1} = 2\Delta t \mu_F \int_\Omega |\mathbf{D}(\mathbf{u}^{n+1})|^2, \quad n = 0, \dots, N-1, \quad i = 1, 2. \tag{2.25}$$

2.5.1 The semi-discretized structure subproblem

In this step $\mathbf{u}$ does not change, so

$$\mathbf{u}^{n+1/2} = \mathbf{u}^n.$$

Furthermore, we define $(\boldsymbol{\eta}^{n+1/2}, \mathbf{v}^{n+1/2}, \mathbf{d}^{n+1/2}, \mathbf{w}^{n+1/2}, \mathbf{k}^{n+1/2}, \mathbf{z}^{n+1/2})$ as the solution of the following problem, written in weak form:

$$\begin{aligned} \rho_K h \int_\omega \frac{\mathbf{v}^{n+1/2} - \mathbf{v}^n}{\Delta t} \cdot \boldsymbol{\psi} R + a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\psi}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{\mathbf{k}_i^{n+1/2} - \mathbf{k}_i^n}{\Delta t} \cdot \boldsymbol{\xi}_i \\ + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{\mathbf{z}_i^{n+1/2} - \mathbf{z}_i^n}{\Delta t} \cdot \boldsymbol{\zeta}_i + a_S(\mathbf{w}^{n+1/2}, \boldsymbol{\zeta}) = 0, \end{aligned} \tag{2.26}$$

$$\int_\omega \frac{\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n}{\Delta t}\cdot\boldsymbol{\psi}R = \int_\omega \mathbf{v}^{n+1/2}\cdot\boldsymbol{\psi}R,$$

$$\left.\begin{aligned}\int_0^{l_i} \frac{\mathbf{d}_i^{n+1/2}-\mathbf{d}_i^n}{\Delta t}\cdot\boldsymbol{\xi}_i &= \int_0^{l_i}\mathbf{k}_i^{n+1/2}\cdot\boldsymbol{\xi}_i,\\ \int_0^{l_i} \frac{\mathbf{w}_i^{n+1/2}-\mathbf{w}_i^n}{\Delta t}\cdot\boldsymbol{\zeta}_i &= \int_0^{l_i}\mathbf{z}_i^{n+1/2}\cdot\boldsymbol{\zeta}_i,\end{aligned}\right\} \quad i=1,\dots,n_E,$$

for all test functions $(\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\in V_{KS}$, where the solution space is defined by:

$$\begin{aligned}V_S^{\Delta t} := \{(\boldsymbol{\eta},\mathbf{v},\mathbf{d},\mathbf{w},\mathbf{k},\mathbf{z}) \in V_K\times L^2(R;\omega)\times V_S\times L^2(\mathcal{N})\times L^2(\mathcal{N}) :\\ \boldsymbol{\eta}\circ\boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E}(0,l_i)\}.\end{aligned} \tag{2.27}$$

Proposition 2.4 For each fixed $\Delta t>0$, the structure subproblem has a unique solution $(\boldsymbol{\eta}^{n+1/2},\mathbf{v}^{n+1/2},\mathbf{d}^{n+1/2},\mathbf{w}^{n+1/2},\mathbf{k}^{n+1/2},\mathbf{z}^{n+1/2})\in V_S^{\Delta t}$.

Proof. The proof is a direct consequence of the Lax-Milgram lemma. To show this, we define a bilinear form on the mesh-shell space by replacing $\mathbf{v}^{n+1/2}$ by $\dfrac{\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n}{\Delta t}$, $\mathbf{k}_i^{n+1/2}$ by $\dfrac{\mathbf{d}_i^{n+1/2}-\mathbf{d}_i^n}{\Delta t}$ and $\mathbf{z}_i^{n+1/2}$ by $\dfrac{\mathbf{w}_i^{n+1/2}-\mathbf{w}_i^n}{\Delta t}$, for $i=1,\dots,n_E$, in the first equation in (2.26). We obtain

$$\begin{aligned}&\rho_K h\int_\omega \frac{\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n}{(\Delta t)^2}\cdot\boldsymbol{\psi}R + a_K(\boldsymbol{\eta}^{n+1/2},\boldsymbol{\psi}) + \rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\frac{\mathbf{d}_i^{n+1/2}-\mathbf{d}_i^n}{(\Delta t)^2}\cdot\boldsymbol{\xi}_i\\ &+\rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\frac{\mathbf{w}_i^{n+1/2}-\mathbf{w}_i^n}{(\Delta t)^2}\cdot\boldsymbol{\zeta}_i + a_S(\mathbf{w}^{n+1/2},\boldsymbol{\zeta})\\ &=\rho_K h\int_\omega\frac{\mathbf{v}^n}{\Delta t}\cdot\boldsymbol{\psi}R + \rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\frac{\mathbf{k}_i^n}{\Delta t}\cdot\boldsymbol{\xi}_i + \rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\frac{\mathbf{z}_i^n}{\Delta t}\cdot\boldsymbol{\zeta}_i.\end{aligned}$$

We multiply this equality by $(\Delta t)^2$ and move all the terms from the $n-$th step to the right-hand side to obtain:

$$\begin{aligned}&\rho_K h\int_\omega \boldsymbol{\eta}^{n+1/2}\cdot\boldsymbol{\psi}R + (\Delta t)^2 a_K(\boldsymbol{\eta}^{n+1/2},\boldsymbol{\psi}) + \rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\mathbf{d}_i^{n+1/2}\cdot\boldsymbol{\xi}_i\\ &+\rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\mathbf{w}_i^{n+1/2}\cdot\boldsymbol{\zeta}_i + (\Delta t)^2 a_S(\mathbf{w}^{n+1/2},\boldsymbol{\zeta}) = \rho_K h\int_\omega\Big(\boldsymbol{\eta}^n+\Delta t\mathbf{v}^n\Big)\cdot\boldsymbol{\psi}R\\ &+\rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\Big(\mathbf{d}_i^n+\Delta t\mathbf{k}_i^n\Big)\cdot\boldsymbol{\xi}_i + \rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\Big(\mathbf{w}_i^n+\Delta t\mathbf{z}_i^n\Big)\cdot\boldsymbol{\zeta}_i.\end{aligned}$$

The left-hand side of the above equation defines the following bilinear form associated with the structure subproblem:

$$a((\boldsymbol{\eta},\mathbf{d},\mathbf{w}),(\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})) := \rho_K h\int_\omega\boldsymbol{\eta}\cdot\boldsymbol{\psi}R + (\Delta t)^2 a_K(\boldsymbol{\eta},\boldsymbol{\psi}) + \rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\mathbf{d}_i\cdot\boldsymbol{\xi}_i$$

$$+\rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{w}_i \cdot \boldsymbol{\zeta}_i + (\Delta t)^2 a_S(\mathbf{w}, \boldsymbol{\zeta}). \tag{2.28}$$

In order to apply the Lax-Milgram lemma we need to prove the continuity and coercivity of the bilinear form (2.28) on V_{KS}. To show that a is coercive, we write

$$\begin{aligned}
& a((\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}), (\boldsymbol{\eta}, \mathbf{d}, \mathbf{w})) \\
& = \rho_K h \int_\omega |\boldsymbol{\eta}|^2 R + (\Delta t)^2 a_K(\boldsymbol{\eta}, \boldsymbol{\eta}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} |\mathbf{d}_i|^2 \\
& + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i |\mathbf{w}_i|^2 + (\Delta t)^2 a_S(\mathbf{w}, \mathbf{w}) \\
& \geq c\Big(\|\boldsymbol{\eta}\|^2_{L^2(R;\omega)} + \|\boldsymbol{\eta}\|^2_k + \|\mathbf{d}\|^2_{L^2(\mathcal{N})} + \|\mathbf{w}\|^2_{L^2(\mathcal{N})} + \|\partial_s \mathbf{w}\|^2_{L^2(\mathcal{N})} \Big) \\
& \geq c\Big(\|\boldsymbol{\eta}\|^2_k + \|\mathbf{d}\|^2_{L^2(\mathcal{N})} + \|\mathbf{w}\|^2_{H^1(\mathcal{N})} \Big).
\end{aligned}$$

Now we use the condition of inextensibility and unshearability to get a bound on $\|\partial_s \mathbf{d}\|^2_{L^2(\mathcal{N})}$:

$$\|\partial_s \mathbf{d}\|_{L^2(\mathcal{N})} = \| -\mathbf{t} \times \mathbf{w}\|_{L^2(\mathcal{N})} \leq C \|\mathbf{w}\|_{L^2(\mathcal{N})}.$$

Notice how the L^2-norm of infinitesimal rotation of cross-sections keeps control over the gradient of displacement of the middle line. This now provides coercivity, i.e.

$$a((\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}), (\boldsymbol{\eta}, \mathbf{d}, \mathbf{w})) \geq c\Big(\|\boldsymbol{\eta}\|^2_k + \|\mathbf{d}\|^2_{H^1(\mathcal{N})} + \|\mathbf{w}\|^2_{H^1(\mathcal{N})} \Big).$$

The Lax-Milgram lemma implies the existence of a unique solution of problem (2.26). □

Proposition 2.5 For each fixed $\Delta t > 0$, the structure subproblem (2.26) satisfies the following discrete energy equality:

$$\begin{aligned}
E_N^{n+1/2} & + \rho_K h \|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2_{L^2(R;\omega)} + a_K(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n) \\
& + \rho_S \|\mathbf{k}^{n+1/2} - \mathbf{k}^n\|^2_a + \rho_S \|\mathbf{z}^{n+1/2} - \mathbf{z}^n\|^2_m \\
& + a_S(\mathbf{w}^{n+1/2} - \mathbf{w}^n, \mathbf{w}^{n+1/2} - \mathbf{w}^n) = E_N^n,
\end{aligned} \tag{2.29}$$

where

$$\|\mathbf{k}\|^2_a := \sum_{i=1}^{n_E} A_i \|\mathbf{k}_i\|_{L^2(0,l_i)}.$$

Proof. We take $(\mathbf{v}^{n+1/2}, \mathbf{k}^{n+1/2}, \mathbf{z}^{n+1/2})$ as a test function in the first equation in (2.26), and replace them with the corresponding expressions: $(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n)/\Delta t$ and $(\mathbf{w}^{n+1/2} - \mathbf{w}^n)/\Delta t$ in the bilinear forms a_S and a_K, respectively, to obtain:

$$\frac{\rho_K h}{\Delta t} \int_\omega (\mathbf{v}^{n+1/2} - \mathbf{v}^n) \cdot \mathbf{v}^{n+1/2} R + \frac{1}{\Delta t} a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n)$$

$$+\frac{\rho_S}{\Delta t}\sum_{i=1}^{n_E} A_i \int_0^{l_i} (\mathbf{k}_i^{n+1/2} - \mathbf{k}_i^n)\cdot\mathbf{k}_i^{n+1/2} + \frac{\rho_S}{\Delta t}\sum_{i=1}^{n_E} \int_0^{l_i} M_i(\mathbf{z}_i^{n+1/2} - \mathbf{z}_i^n)\cdot\mathbf{z}_i^{n+1/2}$$
$$+\frac{1}{\Delta t} a_S(\mathbf{w}^{n+1/2}, \mathbf{w}^{n+1/2} - \mathbf{w}^n) = 0.$$

We then use the algebraic identity $(a-b)\cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$ to deal with the mixed products. After multiplying the entire equation by $2\Delta t$, the first equation in (2.26) can be written as:

$$\begin{aligned}
&\rho_K h(\|\mathbf{v}^{n+1/2}\|^2 + \|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2) + a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\eta}^{n+1/2}) + \\
&a_K(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n) + \rho_S(\|\mathbf{k}^{n+1/2}\|_a^2 + \|\mathbf{k}^{n+1/2} - \mathbf{k}^n\|_a^2) + \\
&\rho_S(\|\mathbf{z}^{n+1/2}\|_m^2 + \|\mathbf{z}^{n+1/2} - \mathbf{z}^n\|_m^2) + a_S(\mathbf{w}^{n+1/2}, \mathbf{w}^{n+1/2}) + a_S(\mathbf{w}^{n+1/2} - \mathbf{w}^n, \\
&\mathbf{w}^{n+1/2} - \mathbf{w}^n) = \rho_K h\|\mathbf{v}^n\|^2 + a_K(\boldsymbol{\eta}^n, \boldsymbol{\eta}^n) + \rho_S\|\mathbf{k}^n\|_a^2 + \rho_S\|\mathbf{z}^n\|_m^2 + a_S(\mathbf{w}^n, \mathbf{w}^n).
\end{aligned}$$

Recall that $\mathbf{u}^{n+1/2} = \mathbf{u}^n$ in this subproblem, so we can add $\rho_F\|\mathbf{u}^{n+1/2}\|^2$ on the left-hand side, and $\rho_F\|\mathbf{u}^n\|^2$ on the right-hand side of the equation, to obtain exactly the energy equality (2.29). □

2.5.2 The semi-discretized fluid subproblem

In this step the shell displacement $\boldsymbol{\eta}$, the mesh displacement $\mathbf{d}$, and the infinitesimal rotation $\mathbf{w}$ do not change, thus:

$$\boldsymbol{\eta}^{n+1} = \boldsymbol{\eta}^{n+1/2}, \quad \mathbf{d}^{n+1} = \mathbf{d}^{n+1/2}, \quad \mathbf{w}^{n+1} = \mathbf{w}^{n+1/2}.$$

Furthermore, the velocity of the mesh displacement and of the infinitesimal rotation have to be zero:

$$\mathbf{k}^{n+1} = \mathbf{k}^{n+1/2}, \quad \mathbf{z}^{n+1} = \mathbf{z}^{n+1/2}.$$

A weak solution of the semi-discretized fluid subproblem is defined to be a function $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ satisfying:

$$\begin{aligned}
&\rho_F \int_\Omega \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}\cdot\boldsymbol{\upsilon} + 2\mu_F \int_\Omega \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\boldsymbol{\upsilon}) \\
&+\rho_K h \int_\omega \frac{\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}}{\Delta t}\cdot\boldsymbol{\psi} R = P_{in}^n \int_{\Gamma_{in}} \upsilon_z - P_{out}^n \int_{\Gamma_{out}} \upsilon_z,
\end{aligned} \tag{2.30}$$

for all test functions $(\boldsymbol{\upsilon}, \boldsymbol{\psi}) \in V_F \times L^2(R;\omega)$ such that $\boldsymbol{\upsilon}|_\Gamma \circ \boldsymbol{\varphi} = \boldsymbol{\psi}$ on ω, where the weak solution space is defined by:

$$V_F^{\Delta t} := \{(\mathbf{u}, \mathbf{v}) \in V_F \times L^2(R;\omega) : \mathbf{u}|_\Gamma \circ \boldsymbol{\varphi} = \mathbf{v} \text{ on } \omega\}. \tag{2.31}$$

The pressure terms are given by $P_{in/out}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)\,dt$.

Proposition 2.6 For each fixed $\Delta t > 0$, the fluid subproblem (2.30) has a unique solution $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1}) \in V_F^{\Delta t}$.

Proof. The proof is again a consequence of the Lax-Milgram lemma. We rewrite the first equation in (2.30) as follows:

$$\begin{aligned}&\frac{\rho_F}{\Delta t}\int_\Omega \mathbf{u}^{n+1}\cdot\boldsymbol{v} + 2\mu_F\int_\Omega \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\boldsymbol{v}) + \frac{\rho_K h}{\Delta t}\int_\Omega \mathbf{v}^{n+1}\cdot\boldsymbol{\psi}R \\ &= \frac{\rho_F}{\Delta t}\int_\Omega \mathbf{u}^n\cdot\boldsymbol{v} + \frac{\rho_K h}{\Delta t}\int_\omega \mathbf{v}^{n+1/2}\cdot\boldsymbol{\psi}R + P_{in}^n\int_{\Gamma_{in}} v_z - P_{out}^n\int_{\Gamma_{out}} v_z.\end{aligned}$$

This defines the following bilinear form associated with problem (2.30):

$$a((\mathbf{u},\mathbf{v}),(\boldsymbol{v},\boldsymbol{\psi})) := \rho_F\int_\Omega \mathbf{u}\cdot\boldsymbol{v} + 2\Delta t\mu_F\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{v}) + \rho_K h\int_\omega \mathbf{v}\cdot\boldsymbol{\psi}R. \tag{2.32}$$

We need to prove that this bilinear form a is coercive and continuous. In order to prove coercivity, we write:

$$\begin{aligned}a((\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v})) &= \rho_F\int_\Omega |\mathbf{u}|^2 + \Delta t 2\mu_F\int_\Omega |\mathbf{D}(\mathbf{u})|^2 + \rho_K h\int_\omega |\mathbf{v}|^2 R \\ &\geq c(\|\mathbf{u}\|^2_{L^2(\Omega)} + \|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)} + \|\mathbf{v}\|^2_{L^2(R;\omega)}) \\ &\geq c(\|\mathbf{u}\|^2_{H^1(\Omega)} + \|\mathbf{v}\|^2_{L^2(R;\omega)}).\end{aligned}$$

By applying Hölder inequality we get the continuity of a:

$$\begin{aligned}a((\mathbf{u},\mathbf{v}),(\boldsymbol{v},\boldsymbol{\psi})) \leq C\Big(&\rho_F\|\mathbf{u}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)} + \Delta t\mu_F\|\mathbf{u}\|_{H^1(\Omega)}\|\boldsymbol{v}\|_{H^1(\Omega)} \\ &+ \rho_K h\|\mathbf{v}\|_{L^2(R;\omega)}\|\boldsymbol{\psi}\|_{L^2(R;\omega)}\Big).\end{aligned}$$

The Lax-Milgram lemma now implies the existence of a unique weak solution $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ of the fluid subproblem (2.30). □

Proposition 2.7 For each fixed $\Delta t > 0$, the solution of problem (2.30) satisfies the following discrete energy inequality:

$$\begin{aligned}&E_N^{n+1} + \rho_F\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2_{L^2(\Omega)} + \rho_K h\|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2_{L^2(R;\omega)} + D_N^{n+1} \\ &\quad\leq E_N^{n+1/2} + C\Delta t((P_{in}^n)^2 + (P_{out}^n)^2).\end{aligned} \tag{2.33}$$

Proof. We begin by replacing the test functions $(\boldsymbol{v}, \boldsymbol{\psi})$ by $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ in the weak formulation (2.30) to obtain:

$$\begin{aligned}&\frac{\rho_F}{\Delta t}\int_\Omega (\mathbf{u}^{n+1} - \mathbf{u}^n)\cdot\mathbf{u}^{n+1} + 2\mu_F\int_\Omega \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\mathbf{u}^{n+1}) \\ &+ \frac{\rho_K h}{\Delta t}\int_\omega (\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2})\cdot\mathbf{v}^{n+1}R = P_{in}^n\int_{\Gamma_{in}} u_z^{n+1} - P_{out}^n\int_{\Gamma_{out}} u_z^{n+1}.\end{aligned}$$

After applying the algebraic identity $(a-b)\cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$ and multiplying the resulting equation by $2\Delta t$ we obtain:

$$
\begin{aligned}
&\rho_F(\|\mathbf{u}^{n+1}\|^2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2) + 2\Delta t \mu_F \|\mathbf{D}(\mathbf{u}^{n+1})\|^2 \\
&+ \rho_K h(\|\mathbf{v}^{n+1}\|^2 + \|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2) \\
&\leq \rho_F \|\mathbf{u}^n\|^2 + \rho_K h \|\mathbf{v}^{n+1/2}\|^2 + C\Delta t((P_{in}^n)^2 + (P_{out}^n)^2).
\end{aligned}
$$

Finally, recall that $\boldsymbol{\eta}^{n+1} = \boldsymbol{\eta}^{n+1/2}$ and $\mathbf{w}^{n+1} = \mathbf{w}^{n+1/2}$ in the fluid subproblem, so we can add $a_K(\boldsymbol{\eta}^{n+1}, \boldsymbol{\eta}^{n+1})$ and $a_S(\mathbf{w}^{n+1}, \mathbf{w}^{n+1})$ on the left-hand side, and $a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\eta}^{n+1/2})$ and $a_S(\mathbf{w}^{n+1/2}, \mathbf{w}^{n+1/2})$ on the right-hand side. Furthermore, since $\mathbf{k}^{n+1} = \mathbf{k}^{n+1/2}$ and $\mathbf{z}^{n+1} = \mathbf{z}^{n+1/2}$, we add $\|\mathbf{k}^{n+1}\|_a^2$ and $\|\mathbf{z}^{n+1}\|_m^2$ on the left-hand side, and $\|\mathbf{k}^{n+1/2}\|_a^2$ and $\|\mathbf{z}^{n+1/2}\|_m^2$ on the right-hand side, to obtain exactly the energy inequality (2.33). □

2.5.3 Uniform energy estimates

Our goal is to ultimately show that there exists a subsequence of functions, parameterized by N (or Δt), defined by the time-discretization via Lie splitting specified above, which converges to a weak solution of Problem 1. To obtain this result, we start by showing that the sequence of approximations defined above, is uniformly bounded (uniformly with respect to Δt) in energy norm.

Lemma 2.8 Let $\Delta t > 0$ and $N = T/\Delta t$. Furthermore, let $E_N^{n+1/2}, E_N^{n+1}$ and D_N^{n+1} be the kinetic energy and dissipation given by (2.24) and (2.25), respectively. Then there exists a constant $K > 0$, independent of Δt (or N) such that the following estimates hold:

1. $E_N^{n+1/2} \leq K, E_N^{n+1} \leq K, \forall n = 0, \dots, N-1,$
2. $\sum_{n=0}^{N-1} D_N^{n+1} \leq K,$
3. $\sum_{n=0}^{N-1} \left(\rho_F \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \rho_K h \|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2 + \rho_K h \|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2\right) \leq K,$
4. $\sum_{n=0}^{N-1} \rho_S\Big(\|\mathbf{k}^{n+1} - \mathbf{k}^n\|_a^2 + \|\mathbf{z}^{n+1} - \mathbf{z}^n\|_m^2\Big) \leq K,$
5. $\sum_{n=0}^{N-1} a_K(\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n) \leq K,$

   $\sum_{n=0}^{N-1} a_S(\mathbf{w}^{n+1} - \mathbf{w}^n, \mathbf{w}^{n+1} - \mathbf{w}^n) \leq K.$

Proof. We begin by adding the energy estimates (2.29) and (2.33) to obtain:

$$
\begin{aligned}
&E_N^{n+1/2} + \rho_K h \|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2 + a_K(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n) \\
&+ \rho_S \|\mathbf{k}^{n+1/2} - \mathbf{k}^n\|_a^2 + \rho_S \|\mathbf{z}^{n+1/2} - \mathbf{z}^n\|_m^2 \\
&+ a_S(\mathbf{w}^{n+1/2} - \mathbf{w}^n, \mathbf{w}^{n+1/2} - \mathbf{w}^n) + E_N^{n+1} + \rho_F \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 \\
&+ \rho_K h \|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2 + D_N^{n+1} \leq E_N^n + E_N^{n+1/2} + C\Delta t((P_{in}^n)^2 + (P_{out}^n)^2).
\end{aligned}
$$

We take the sum from $n = 0$ to $n = N - 1$ on both sides to obtain:

$$
\begin{aligned}
&E_N^N + \sum_{n=0}^{N-1} D_N^{n+1} + \sum_{n=0}^{N-1} \Big(\rho_F \|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \rho_K h \|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2 \\
&+ \rho_K h \|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2 + \rho_S \|\mathbf{k}^{n+1/2} - \mathbf{k}^n\|_a^2 + \rho_S \|\mathbf{z}^{n+1/2} - \mathbf{z}^n\|_m^2 \\
&+ a_K(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n) + a_S(\mathbf{w}^{n+1/2} - \mathbf{w}^n, \mathbf{w}^{n+1/2} - \mathbf{w}^n)\Big) \\
&\leq E_0 + C\Delta t \sum_{n=0}^{N-1} \left((P_{in}^n)^2 + (P_{out}^n)^2\right).
\end{aligned}
$$

The term involving the inlet and outlet pressure data can be easily estimated by recalling that on each subinterval (t_n, t_{n+1}) the pressure data is approximated by a constant, which is equal to the average value of the pressure over that time interval. Therefore, after using Hölder inequality, we have:

$$
\Delta t \sum_{n=0}^{N-1} (P_{in}^n)^2 = \Delta t \sum_{n=0}^{N-1} \left(\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in}(t) dt \right)^2 \leq \|P_{in}\|^2_{L^2(0,T)}. \tag{2.34}
$$

We can now bound the right-hand side in the above energy estimate by using the just calculated pressure estimate, to obtain all the statements in the Lemma, with the constant $K = E_0 + C\Delta t \left(\|P_{in}\|^2_{L^2(0,T)} + \|P_{out}\|^2_{L^2(0,T)}\right)$. □

2.6 Convergence of approximate solutions

The approach described above defines a set of discrete values in time, which can be used to define approximate functions on $(0, T)$. Indeed, we define approximate solutions on $(0, T)$ to be the functions, which are piecewise constant on each subinterval $((n - 1)\Delta t, n\Delta t],\ n = 1, \dots, N$ of $(0, T)$:

- $\mathbf{u}_N(t, \cdot) = \mathbf{u}_N^n,\ \boldsymbol{\eta}_N(t, \cdot) = \boldsymbol{\eta}_N^n, \forall t \in ((n - 1)\Delta t, n\Delta t]$,
- $\mathbf{v}_N(t, \cdot) = \mathbf{v}_N^n,\ \hat{\mathbf{v}}_N(t, \cdot) = \mathbf{v}_N^{n-1/2}, \forall t \in ((n - 1)\Delta t, n\Delta t]$,
- $\mathbf{d}_N(t, \cdot) = \mathbf{d}_N^n,\ \mathbf{w}_N(t, \cdot) = \mathbf{w}_N^n, \forall t \in ((n - 1)\Delta t, n\Delta t]$,
- $\mathbf{k}_N(t, \cdot) = \mathbf{k}_N^n,\ \mathbf{z}_N(t, \cdot) = \mathbf{z}_N^n, \forall t \in ((n - 1)\Delta t, n\Delta t]$.

In the second bullet above we used $\hat{\mathbf{v}}_N(t, \cdot) = \mathbf{v}_N^{n-1/2}$ to denote the approximate shell velocity functions determined by the structure subproblem, and $\mathbf{v}_N(t, \cdot) = \mathbf{v}_N^n$ to denote the approximate shell velocity functions determined by the fluid subproblem. We emphasize that these are not necessarily the same. As a consequence, the kinematic coupling condition, which involves the shell velocity, is asynchronously satisfied by this scheme at each time step. We will show, however, that the difference between these approximate sequences for the shell velocity converges to zero in L^2 as $\Delta t \to 0$.

2.6.1 Weak and weak* convergence

Using the energy estimates presented in Lemma 2.8, we will show that the approximate sequences of functions defined above for all $t \in (0,T)$, are uniformly bounded in the appropriate solution spaces involving both space and time. This will provide weakly and weakly* convergent subsequences of approximate functions, for which we will show convergence to a weak solution of the coupled, continuous problem, as $\Delta t \to 0$.

Proposition 2.9 The sequence $(\mathbf{u}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ $\cap\, L^2(0,T;H^1(\Omega))$.

Proof. The uniform boundedness of $(\mathbf{u}_N)$ in $L^\infty(0,T;L^2(\Omega))$ follows directly from the first statement of Lemma 2.8. To show the uniform boundedness of $(\mathbf{u}_N)$ in $L^2(0,T;H^1(\Omega))$, notice that from the second statement of Lemma 2.8 we have:

$$\sum_{n=1}^{N} \int_\Omega |\mathbf{D}(\mathbf{u}_N^n)|^2 \Delta t \leq K, \tag{2.35}$$

where $\mathbf{D}(\mathbf{u}_N^n)$ is the symmetrized gradient. By applying Korn inequality we obtain

$$\|\nabla \mathbf{u}_N^n\|^2_{L^2(\Omega)} \leq C\|\mathbf{D}(\mathbf{u}_N^n)\|^2_{L^2(\Omega)}.$$

Taking the sum from $n = 1, \ldots, N$, we get the following estimate

$$\sum_{n=1}^{N} \|\nabla \mathbf{u}_N^n\|^2_{L^2(\Omega)} \Delta t \leq C \sum_{n=1}^{N} \|\mathbf{D}(\mathbf{u}_N^n)\|^2_{L^2(\Omega)} \Delta t,$$

which implies that the sequence $(\nabla \mathbf{u}_N)$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$, and so the sequence $(\mathbf{u}_N)$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. □

Proposition 2.10 The sequence $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;V_K)$, and the sequence $(\mathbf{w}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H^1(\mathcal{N}))$.

Proof. From Lemma 2.8 we have that $E_N^n \leq K$, $\forall n = 0, \ldots, N-1$, which implies

$$\|\boldsymbol{\eta}_N(t)\|_k^2 \leq a_K(\boldsymbol{\eta}_N(t), \boldsymbol{\eta}_N(t)) \leq K, \quad \forall t \in [0,T].$$

Therefore,

$$\|\boldsymbol{\eta}_N\|_{L^\infty(0,T;V_K)} \leq K.$$

The boundedness of the sequence $(\mathbf{w}_N)_{N\in\mathbb{N}}$ also follows from the first statement of Lemma 2.8. Namely, we have

$$\|\mathbf{w}_N(t)\|^2_{L^2(\mathcal{N})} \leq \|\mathbf{w}_N(t)\|^2_m \leq K,$$
$$\|\partial_s \mathbf{w}_N(t)\|^2_{L^2(\mathcal{N})} \leq a_S(\mathbf{w}_N(t), \mathbf{w}_N(t)) \leq K,$$

which concludes the proof. □

The following uniform bounds for the shell and mesh approximate velocities are a direct consequence of Lemma 2.8.

Proposition 2.11 The following uniform bounds for the shell and mesh approximate velocities hold:

(i) $(\mathbf{v}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(R;\omega))$,
$(\hat{\mathbf{v}}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(R;\omega))$,

(ii) $(\mathbf{k}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathcal{N}))$,
$(\mathbf{z}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathcal{N}))$.

To pass to the limit in the weak formulation of approximate solutions, we need additional regularity in time of the sequences $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}, (\mathbf{d}_N)_{N\in\mathbb{N}}$ and $(\mathbf{w}_N)_{N\in\mathbb{N}}$. For this purpose, we introduce a slightly different set of approximate functions. Namely, for each fixed Δt, define $\tilde{\boldsymbol{\eta}}_N, \tilde{\mathbf{d}}_N$ and $\tilde{\mathbf{w}}_N$ to be continuous, *linear* on each subinterval $[(n-1)\Delta t, n\Delta t],\ n = 1,\ldots,N$, and such that

$$\begin{aligned}
&\tilde{\mathbf{u}}_N(n\Delta t,\cdot) = \mathbf{u}_N(n\Delta t,\cdot),\\
&\tilde{\boldsymbol{\eta}}_N(n\Delta t,\cdot) = \boldsymbol{\eta}_N(n\Delta t,\cdot), \tilde{\mathbf{v}}_N(n\Delta t,\cdot) = \mathbf{v}_N(n\Delta t,\cdot),\\
&\tilde{\mathbf{d}}_N(n\Delta t,\cdot) = \mathbf{d}_N(n\Delta t,\cdot), \tilde{\mathbf{w}}_N(n\Delta t,\cdot) = \mathbf{w}_N(n\Delta t,\cdot),\\
&\tilde{\mathbf{k}}_N(n\Delta t,\cdot) = \mathbf{k}_N(n\Delta t,\cdot), \tilde{\mathbf{z}}_N(n\Delta t,\cdot) = \mathbf{z}_N(n\Delta t,\cdot).
\end{aligned} \tag{2.36}$$

We now observe:

$$\partial_t\tilde{\boldsymbol{\eta}}_N(t) = \frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t} = \frac{\boldsymbol{\eta}_N^{n+1/2} - \boldsymbol{\eta}_N^n}{\Delta t} = \mathbf{v}_N^{n+1/2}, \quad t \in (n\Delta t, (n+1)\Delta t].$$

Since $\hat{\mathbf{v}}_N$ is a piecewise constant function, as defined before via $\hat{\mathbf{v}}_N(t,\cdot) = \mathbf{v}_N^{n+1/2}$, for $t \in (n\Delta t, (n+1)\Delta t]$, we see that

$$\partial_t\tilde{\boldsymbol{\eta}}_N = \hat{\mathbf{v}}_N \text{ a.e. on } (0,T). \tag{2.37}$$

From (2.37), and from the uniform boundedness of $E_N^{n+i/2}$ provided by Lemma 2.8, we obtain the uniform boundedness of $(\tilde{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ in $W^{1,\infty}(0,T;L^2(R;\omega))$. Now, since sequences $(\tilde{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ and $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ have the same limit (distributional limit is unique), we get that the weak* limit of $\boldsymbol{\eta}_N$ is, in fact, in $W^{1,\infty}(0,T;L^2(R;\omega))$.

Using analogous arguments, one also obtains that the weak* limits of $(\mathbf{d}_N)_{N\in\mathbb{N}}$ and $(\mathbf{w}_N)_{N\in\mathbb{N}}$ are in $W^{1,\infty}(0,T;L^2(\mathcal{N}))$. This is because the corresponding velocity approximations are uniformly bounded in the corresponding norms, as stated in part 4. of Lemma 2.8.

Notice that we do not get any bounds on the sequence $(\mathbf{d}_N)_{N\in\mathbb{N}}$ from the uniform energy estimates. Nevertheless, using the condition of inextensibility and unshearability, together with the regularity of $\mathbf{w}_N$, one can easily prove the H^1-regularity in space of $\mathbf{d}_N$. More precisely, the following result holds true:

Corollary 2.12 The sequence $(\mathbf{d}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H^1(\mathcal{N}))$.

From the uniform boundedness of approximate sequences we can now conclude the following weak and weak* convergence results:

Lemma 2.13 There exist subsequences $(\mathbf{u}_N)_{N\in\mathbb{N}}, (\boldsymbol{\eta}_N)_{N\in\mathbb{N}}, (\mathbf{v}_N)_{N\in\mathbb{N}}, (\hat{\mathbf{v}}_N)_{N\in\mathbb{N}}$, $(\mathbf{d}_N)_{N\in\mathbb{N}}, (\mathbf{w}_N)_{N\in\mathbb{N}}, (\mathbf{k}_N)_{N\in\mathbb{N}}, (\mathbf{z}_N)_{N\in\mathbb{N}}$, and the functions $\mathbf{u}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$, $\boldsymbol{\eta}\in L^\infty(0,T;V_K)\cap W^{1,\infty}(0,T;L^2(R;\omega))$, $\mathbf{d},\mathbf{w}\in L^\infty(0,T;H^1(\mathcal{N}))\cap W^{1,\infty}(0,T;L^2(\mathcal{N}))$, $\mathbf{v},\hat{\mathbf{v}}\in L^\infty(0,T;L^2(R;\omega))$, and $\mathbf{k},\mathbf{z}\in L^\infty(0,T;L^2(\mathcal{N}))$, such that

$$
\begin{aligned}
&\mathbf{u}_N \rightharpoonup \mathbf{u} \text{ weakly* in } L^\infty(0,T;L^2(\Omega)),\\
&\mathbf{u}_N \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0,T;H^1(\Omega)),\\
&\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta} \text{ weakly* in } L^\infty(0,T;V_K),\\
&\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta} \text{ weakly* in } W^{1,\infty}(0,T;L^2(R;\omega)),\\
&\mathbf{d}_N \rightharpoonup \mathbf{d} \text{ weakly* in } L^\infty(0,T;H^1(\mathcal{N})),\\
&\mathbf{d}_N \rightharpoonup \mathbf{d} \text{ weakly* in } W^{1,\infty}(0,T;L^2(\mathcal{N})),\\
&\mathbf{w}_N \rightharpoonup \mathbf{w} \text{ weakly* in } L^\infty(0,T;H^1(\mathcal{N})),\\
&\mathbf{w}_N \rightharpoonup \mathbf{w} \text{ weakly* in } W^{1,\infty}(0,T;L^2(\mathcal{N})),\\
&\mathbf{v}_N \rightharpoonup \mathbf{v} \text{ weakly* in } L^\infty(0,T;L^2(R;\omega)),\\
&\hat{\mathbf{v}}_N \rightharpoonup \hat{\mathbf{v}} \text{ weakly* in } L^\infty(0,T;L^2(R;\omega)),\\
&\mathbf{k}_N \rightharpoonup \mathbf{k} \text{ weakly* in } L^\infty(0,T;L^2(\mathcal{N})),\\
&\mathbf{z}_N \rightharpoonup \mathbf{z} \text{ weakly* in } L^\infty(0,T;L^2(\mathcal{N})).
\end{aligned}
$$

Furthermore,

$$\mathbf{v}=\hat{\mathbf{v}}.$$

Proof. We only need to show that $\mathbf{v}=\hat{\mathbf{v}}$. To show this, we use the definition of approximate sequences as step functions in t, i.e.

$$
\begin{aligned}
\|\mathbf{v}_N-\hat{\mathbf{v}}_N\|^2_{L^2(0,T;L^2(R;\omega))} &= \int_0^T \|\mathbf{v}_N-\hat{\mathbf{v}}_N\|^2_{L^2(R;\omega)}dt\\
&= \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \|\mathbf{v}_N^{n+1}-\mathbf{v}_N^{n+1/2}\|^2_{L^2(R;\omega)}dt\\
&= \sum_{n=0}^{N-1} \|\mathbf{v}_N^{n+1}-\mathbf{v}_N^{n+1/2}\|^2_{L^2(R;\omega)}\Delta t \le K\Delta t.
\end{aligned}
$$

The last inequality follows from the third statement of Lemma 2.8. By letting $\Delta t \to 0$, we get that $\mathbf{v} = \hat{\mathbf{v}}$. $\square$

2.6.2 Passing to the limit and proof of main result

We start by first writing the weak formulation of the coupled, semi-discretized problem. For this purpose take $(\boldsymbol{\psi}(t), \boldsymbol{\xi}(t), \boldsymbol{\zeta}(t))$ as the test functions in the first equation in (2.26), and integrate with respect to t from $n\Delta t$ to $(n+1)\Delta t$. Then, take $(\boldsymbol{\upsilon}(t), \boldsymbol{\psi}(t))$ as the test functions in the first equation in (2.30), and again integrate over the same time interval. Add the two equations together to obtain

$$\begin{aligned}
&\rho_F \int_{n\Delta t}^{(n+1)\Delta t} \int_\Omega \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \boldsymbol{\upsilon} + 2\mu_F \int_{n\Delta t}^{(n+1)\Delta t} \int_\Omega \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\boldsymbol{\upsilon}) \\
&+ \rho_K h \int_{n\Delta t}^{(n+1)\Delta t} \int_\omega \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} \cdot \boldsymbol{\psi} R + \int_{n\Delta t}^{(n+1)\Delta t} a_K(\boldsymbol{\eta}^{n+1}, \boldsymbol{\psi}) \\
&+ \rho_S \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{\mathbf{k}_i^{n+1} - \mathbf{k}_i^n}{\Delta t} \cdot \boldsymbol{\xi}_i + \rho_S \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{\mathbf{z}_i^{n+1} - \mathbf{z}_i^n}{\Delta t} \cdot \boldsymbol{\zeta}_i \\
&+ \int_{n\Delta t}^{(n+1)\Delta t} a_S(\mathbf{w}^{n+1}, \boldsymbol{\zeta}) = \int_{n\Delta t}^{(n+1)\Delta t} P_{in}^n \int_{\Gamma_{in}} \upsilon_z - \int_{n\Delta t}^{(n+1)\Delta t} P_{out}^n \int_{\Gamma_{out}} \upsilon_z.
\end{aligned}$$

By using the definition of approximate solutions as functions of t, and by taking the sum from $n = 0, \dots, N-1$ to get the time integrals over $(0, T)$, we get:

$$\begin{aligned}
&\rho_F \int_0^T \int_\Omega \partial_t \tilde{\mathbf{u}}_N \cdot \boldsymbol{\upsilon} + 2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}_N) : \mathbf{D}(\boldsymbol{\upsilon}) \\
&+ \rho_K h \int_0^T \int_\omega \partial_t \tilde{\mathbf{v}}_N \cdot \boldsymbol{\psi} R + \int_0^T a_K(\boldsymbol{\eta}_N, \boldsymbol{\psi}) \\
&+ \rho_S \int_0^T \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t (\tilde{\mathbf{k}}_N)_i \cdot \boldsymbol{\xi}_i + \rho_S \int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t (\tilde{\mathbf{z}}_N)_i \cdot \boldsymbol{\zeta}_i \\
&+ \int_0^T a_S(\mathbf{w}_N, \boldsymbol{\zeta}) = \int_0^T P_{in}^N \int_{\Gamma_{in}} \upsilon_z - \int_0^T P_{out}^N \int_{\Gamma_{out}} \upsilon_z.
\end{aligned}$$

Here, $\tilde{\mathbf{u}}_N, \tilde{\mathbf{v}}_N, \tilde{\mathbf{k}}_N$ and $\tilde{\mathbf{z}}_N$ are the piecewise linear functions defined in (2.36), while $\mathbf{u}_N, \boldsymbol{\eta}_N$ and $\mathbf{w}_N$ are piecewise constant functions. Integration by parts with respect to time gives:

$$\begin{aligned}
&- \rho_F \int_0^T \int_\Omega \tilde{\mathbf{u}}_N \cdot \partial_t \boldsymbol{\upsilon} + 2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}_N) : \mathbf{D}(\boldsymbol{\upsilon}) - \rho_K h \int_0^T \int_\omega \tilde{\mathbf{v}}_N \cdot \partial_t \boldsymbol{\psi} R \\
&+ \int_0^T a_K(\boldsymbol{\eta}_N, \boldsymbol{\psi}) - \rho_S \int_0^T \sum_{i=1}^{n_E} A_i \int_0^{l_i} (\tilde{\mathbf{k}}_N)_i \cdot \partial_t \boldsymbol{\xi}_i \\
&- \rho_S \int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} M_i (\mathbf{z}_N)_i \cdot \partial_t \boldsymbol{\zeta}_i + \int_0^T a_S(\mathbf{w}_N, \boldsymbol{\zeta}) = \int_0^T P_{in}^N \int_{\Gamma_{in}} \upsilon_z \qquad (2.38) \\
&- \int_0^T P_{out}^N \int_{\Gamma_{out}} \upsilon_z + \rho_F \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\upsilon}(0) + \rho_K h \int_\omega \mathbf{v}_0 \cdot \boldsymbol{\psi}(0) R
\end{aligned}$$

$$+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{k}_{0i} \cdot \boldsymbol{\xi}_i(0) + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{z}_{0i} \cdot \boldsymbol{\zeta}_i(0),$$

where we recall that

$$\nabla \cdot \mathbf{u}_N = 0, \quad \mathbf{u}_N|_\Gamma \circ \varphi = \mathbf{v}_N, \quad \boldsymbol{\eta}_N \circ \boldsymbol{\pi} = \mathbf{d}_N.$$

Using the convergence results obtained for the approximate functions in Lemma 2.13, we can pass to the limit in all the terms. Thus, we have shown that the limiting functions satisfy the weak form of Problem 1 in the sense of Definition 2.2. This completes the proof of the main result of this chapter, stated in Theorem 2.3.

Chapter 3

Nonlinear, moving-boundary fluid-mesh-shell interaction problem

3.1 Model description

3.1.1 The fluid

We consider the flow of an incompressible, viscous fluid, modeled by the Navier-Stokes equations, in a three-dimensional time-dependent cylindrical domain of reference length L and reference radius R. The reference fluid domain will be denoted by Ω. The boundary of the cylindrical domain consists of three parts: the lateral boundary, whose location is not known a priori but depends on the motion of the fluid occupying the domain, the inlet boundary Γ_{in} and the outlet boundary Γ_{out}. The elastodynamics of the lateral boundary of the fluid domain is modeled by the linearly elastic Koiter shell equations coupled with a system of linearly elastic one-dimensional curved rod equations describing the motion of a mesh-like structure. The reference location of the lateral boundary will be denoted by Γ. We will be assuming that the lateral boundary displacement is non-negligible in all three spatial directions and is given by a function $\boldsymbol{\eta} : (0,T) \times (0,L) \times (0,2\pi) \to \mathbb{R}^2$ with

$$\boldsymbol{\eta}(t,z,\theta) = (\eta_z(t,z,\theta), \eta_r(t,z,\theta), \eta_\theta(t,z,\theta)).$$

The time-dependent deformation of the fluid domain determined by the interaction between fluid flow and the elastic (lateral) part of the domain boundary is defined as an arbitrary, injective and orientation preserving mapping $\boldsymbol{\phi}^{\boldsymbol{\eta}}(t,\cdot) : \Omega \to \mathbb{R}^3, t \in (0,T)$, such that

$$\boldsymbol{\phi}^{\boldsymbol{\eta}}(t)|_\Gamma = \mathbf{id} + \boldsymbol{\eta}(t,z,\theta).$$

We denote by $\Omega^{\boldsymbol{\eta}}(t) = \boldsymbol{\phi}^{\boldsymbol{\eta}}(t,\Omega)$ the deformed fluid domain at time t and by $\Gamma^{\boldsymbol{\eta}}(t) = \boldsymbol{\phi}^{\boldsymbol{\eta}}(t,\Gamma)$ the corresponding deformed lateral boundary. Notice that we included superscript $\boldsymbol{\eta}$ in the notation in order to emphasize the dependence of the position of the fluid domain, as well as the lateral boundary, on the displacement $\boldsymbol{\eta}$, which is one of the unknowns in the problem.

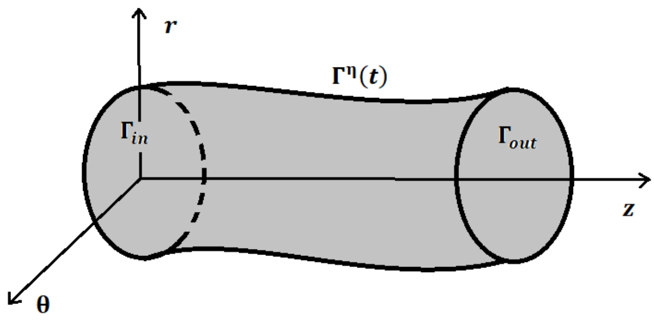

Figure 3.1: The fluid domain

We are interested in studying a pressure-driven flow through $\Omega^{\boldsymbol{\eta}}(t)$ of an incompressible, viscous, Newtonian fluid modeled by the Navier-Stokes equations

$$\left.\begin{array}{rcl} \rho_F\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) & = & \nabla\cdot\boldsymbol{\sigma}, \\ \nabla\cdot\mathbf{u} & = & 0, \end{array}\right\} \text{ in } \Omega^{\boldsymbol{\eta}}(t), t\in(0,T), \tag{3.1}$$

where ρ_F denotes the fluid density, $\mathbf{u}$ is the fluid velocity, $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu_F\mathbf{D}(\mathbf{u})$ is the fluid Cauchy stress tensor, p is the fluid pressure, μ_F is the dynamic viscosity coefficient, and $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$. At the inlet and outlet boundaries, we prescribe zero tangential velocity and dynamic pressure ([33]):

$$\left.\begin{array}{rcl} p + \dfrac{\rho_F}{2}|\mathbf{u}|^2 & = & P_{in/out}(t), \\ \mathbf{u}\times\mathbf{e}_z & = & 0, \end{array}\right\} \text{ on } \Gamma_{in/out}, \tag{3.2}$$

where $P_{in/out}$ are given. Therefore, the fluid flow is driven by a prescribed dynamic pressure drop, and the flow enters and leaves the fluid domain orthogonally to the inlet and outlet boundary.

3.1.2 The shell

The elastodynamics of the lateral boundary of the fluid domain will be modeled by the linearly elastic cylindrical Koiter shell equations ([52]) capturing displacement in all three spatial directions. The shell thickness will be denoted by $h > 0$, the length by L and its reference radius of the middle surface by R. Furthermore, we assume that the cylindrical shell is clamped at its end points. This reference configuration, which we denote by Γ, can be parameterized by

$$\boldsymbol{\varphi} : \omega \to \mathbb{R}^3, \quad \boldsymbol{\varphi}(z,\theta) = (z, R\cos\theta, R\sin\theta),$$

where $\omega = (0, L) \times (0, 2\pi)$, and $R > 0$. Therefore, the reference configuration is given by

$$\Gamma = \{(z, R\cos\theta, R\sin\theta) : z \in (0, L), \theta \in (0, 2\pi)\}.$$

The first fundamental form of the cylinder Γ, or the metric tensor in covariant A_c or contravariant A^c components are respectively given by

$$A_c = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \quad A^c = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix},$$

and the area element is $dS = \sqrt{\det A_c}\; dzd\theta = R\; dzd\theta$. The second fundamental form of the cylinder Γ, or the curvature tensor in covariant components is given by

$$B_c = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}.$$

Under the action of force, the Koiter shell is displaced from its reference configuration Γ by a displacement $\boldsymbol{\eta} = \boldsymbol{\eta}(t, z, \theta) = (\eta_z, \eta_r, \eta_\theta)$, where η_z, η_r and η_θ denote the tangential, radial and azimuthal components of the displacement.

The cylindrical Koiter shell is assumed to be clamped at the end points, giving rise to the following boundary conditions:

$$\boldsymbol{\eta}(t, 0, \theta) = \boldsymbol{\eta}(t, L, \theta) = 0, \ \theta \in (0, 2\pi),$$
$$\partial_z\eta_r(t, 0, \theta) = \partial_z\eta_r(t, L, \theta) = 0, \ \theta \in (0, 2\pi),$$

and at $\theta \in 0, 2\pi$, we assume periodic boundary conditions:

$$\boldsymbol{\eta}(t, z, 0) = \boldsymbol{\eta}(t, z, 2\pi), \ z \in (0, L),$$
$$\partial_\theta\eta_r(t, z, 0) = \partial_\theta\eta_r(t, z, 2\pi), \ z \in (0, L).$$

Deformation of a given Koiter shell depends on its elastic properties. The elastic properties of our cylindrical Koiter shell are defined by the following elasticity tensor $\mathcal{A}$:

$$\mathcal{A}E = \frac{4\lambda\mu}{\lambda + 2\mu}(A^c \cdot E)A^c + 4\mu A^c E A^c, \quad E \in \mathrm{Sym}(\mathbb{R}^2),$$

where μ and λ are Lamé constants of the elastic material constituting the shell. A Koiter shell can undergo stretching of the middle surface and flexure. Stretching of the middle surface is measured by the change of metric tensor, while flexure is measured by the change of curvature tensor. The linearized change of metric tensor $\boldsymbol{\gamma}$ and the linearized change of curvature tensor $\boldsymbol{\varrho}$ are given by the following matrices

$$\boldsymbol{\gamma}(\boldsymbol{\eta}) = \begin{pmatrix} \partial_z \eta_z & \frac{1}{2}(\partial_\theta \eta_z + R\partial_z \eta_\theta) \\ \frac{1}{2}(\partial_\theta \eta_z + R\partial_z \eta_\theta) & R\partial_\theta \eta_\theta + R\eta_r \end{pmatrix},$$

$$\boldsymbol{\varrho}(\boldsymbol{\eta}) = \begin{pmatrix} -\partial_{zz}\eta_r & -\partial_{z\theta}\eta_r + \partial_z \eta_\theta \\ -\partial_{z\theta}\eta_r + \partial_z \eta_\theta & -\partial_{\theta\theta}\eta_r + 2\partial_\theta \eta_\theta + \eta_r \end{pmatrix}.$$

With the corresponding change of metric and change of curvature tensors, we can write the elastic energy of the deformed shell:

$$E(\boldsymbol{\eta}) = \frac{h}{4}\int_\omega \mathcal{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\eta})R + \frac{h^3}{48}\int_\omega \mathcal{A}\boldsymbol{\varrho}(\boldsymbol{\eta}) : \boldsymbol{\varrho}(\boldsymbol{\eta})R.$$

Let V_K denote the space of all H^2 functions which satisfy prescribed boundary conditions, i.e.

$$\begin{aligned} V_K = \{\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in H^2(\omega) \times H^2(\omega) \times H^2(\omega) : \\ \boldsymbol{\eta}(t,z,\theta) = \partial_z \eta_r(t,z,\theta) = 0, z \in \{0, L\}, \theta \in (0, 2\pi) \\ \boldsymbol{\eta}(t,z,0) = \boldsymbol{\eta}(t,z,2\pi), \partial_\theta \eta_r(t,z,0) = \partial_\theta \eta_r(t,z,2\pi), z \in (0,L)\}, \end{aligned} \tag{3.3}$$

equipped with the corresponding norm

$$\|\boldsymbol{\eta}\|^2_{H^2(\omega)} := \|\boldsymbol{\eta}\|^2_{H^2(\omega;\mathbb{R}^3)} = \|\eta_z\|^2_{H^2(\omega)} + \|\eta_r\|^2_{H^2(\omega)} + \|\eta_\theta\|^2_{H^2(\omega)}.$$

Given a force $\mathbf{f}$, the loaded shell deforms under the applied force, and the displacement $\boldsymbol{\eta}$ is a solution to the following elastodynamics problem, written in weak form: find $\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in V_K$

$$\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \boldsymbol{\mathcal{L}}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle = \int_\omega \mathbf{f} \cdot \boldsymbol{\psi} R, \ \forall \boldsymbol{\psi} \in V_K. \tag{3.4}$$

Here, ρ_K is the shell density and $\boldsymbol{\mathcal{L}}$ is an operator that describes elastic properties (change of metric tensor and change of curvature tensor) of the shell and includes the regularization term (with the regularization parameter ε_K):

$$\begin{aligned} \langle \boldsymbol{\mathcal{L}}\boldsymbol{\eta}, \boldsymbol{\psi}\rangle = & h\int_\omega \mathcal{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\psi})R + \frac{h^3}{12}\int_\omega \mathcal{A}\boldsymbol{\varrho}(\boldsymbol{\eta}) : \boldsymbol{\varrho}(\boldsymbol{\psi})R \\ & + \varepsilon_K \int_\omega (\Delta\eta_z \Delta\psi_z + \Delta\eta_\theta \Delta\psi_\theta)R. \end{aligned}$$

By using the same procedure as in the proof of Theorem 2.6-4 in [30] (inequality of Korn's type on a general surface), we can easily get the coercivity of the operator $\boldsymbol{\mathcal{L}}$, i.e.

$\langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\eta}\rangle \geq c\|\boldsymbol{\eta}\|^2_{H^2(\omega)}, \forall \boldsymbol{\eta} \in V_K$.

Remark We would like to draw the attention of the reader to the fact that the bound on the H^2-norm in the tangential and azimuthal direction was made possible by the regularization term.

The differential formulation of the shell elastodynamics problem on $(0,T)\times\omega$ is given by:

$$\rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} = \mathbf{f} R. \tag{3.5}$$

Justification of a Koiter shell model can be found in [32].

3.1.3 The elastic mesh

An elastic mesh is a three-dimensional elastic body defined as a union of three-dimensional slender components called struts. Since each strut is ”thin”, meaning that its height and width are small comparing to its length, one-dimensional curved rod model can be used to approximate strut's elastodynamic properties. The $i-$th curved rod is parameterized via

$$\mathbf{P}_i : [0, l_i] \to \varphi(\overline{\omega}), \quad i = 1, \dots, n_E,$$

keeping in mind that one spatial dimension corresponds to the parameterization of the middle line of the curved rod. Here, n_E denotes the number of the curved rods in a mesh.

By using $s \in (0, l_i)$ to denote the location along the middle line, and $\mathbf{d}_i(t,s)$ to denote the displacement of the middle line from its reference configuration, $\mathbf{w}_i(t,s)$ the infinitesimal rotation of cross-sections, $\mathbf{q}_i(t,s)$ the contact moment, and $\mathbf{p}_i(t,s)$ the contact force, the following system of equations will be used to model the elastodynamics of 1D curved rods:

$$\begin{array}{rcl} \rho_S A_i \partial_t^2 \mathbf{d}_i &=& \partial_s \mathbf{p}_i + \mathbf{f}_i, \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &=& \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i, \\ 0 &=& \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i, \\ 0 &=& \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i. \end{array} \tag{3.6}$$

Here, ρ_S is the strut's material density, A_i is the cross-sectional area of the $i-$th rod, M_i is the matrix related to the moments of inertia of the cross-sections, $\mathbf{f}_i$ is the line force density acting on the $i-$th rod, and $\mathbf{t}_i$ is the unit tangent on the middle line of the $i-$th rod. Matrix H_i is a positive definite matrix which describes the elastic properties and the geometry of the cross section, while matrix $Q_i \in SO(3)$ represents the local basis at each point of the middle line of the $i-$th rod (see [2] for more details).

The first two equations describe the linear impulse-momentum law and the angular impulse-momentum law, respectively, while the last two equations describe a constitutive relation for a curved, linearly elastic rod, and the condition of inextensibility and unshearability, respectively.

System (3.6) is defined on a graph domain, determined by the geometry and topology of the mesh structure. Let us denote by $\mathcal{V}$ a set of graph vertices (points where the middle lines meet), and by $\mathcal{E}$ a set of edges (pairing of vertices). The ordered pair $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ defines the mesh net topology. At each vertex $V \in \mathcal{V}$, the following coupling conditions are enforced:

- kinematic coupling conditions describing continuity of displacements and infinitesimal rotations,
- dynamic coupling conditions describing the balance of contact forces and contact moments.

We are interested in weak solutions to the 1D mesh net problem, i.e., to the problem consisting of all the functions satisfying the system of linear equations (3.6) on a graph domain, with the above-mentioned coupling conditions holding at graph's vertices. To define the weak solution space, we first introduce a function space consisting of all the H^1-functions $(\mathbf{d}, \mathbf{w})$ defined on the entire net $\mathcal{N}$, such that they satisfy the kinematic coupling conditions at each vertex $V \in \mathcal{V}$:

$$\begin{aligned} H^1(\mathcal{N}; \mathbb{R}^6) = \{(\mathbf{d}, \mathbf{w}) = ((\mathbf{d}_1, \mathbf{w}_1), \dots, (\mathbf{d}_{n_E}, \mathbf{w}_{n_E})) \in \prod_{i=1}^{n_E} H^1(0, l_i; \mathbb{R}^6) :& \\ \mathbf{d}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{d}_j(\mathbf{P}_j^{-1}(V)), \mathbf{w}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{w}_j(\mathbf{P}_j^{-1}(V)),& \\ \forall V \in \mathcal{V}, V = e_i \cap e_j,\ i, j = 1, \dots, n_E\}.& \end{aligned}$$

The solution space is defined to contain the conditions of inextensibility and unshearability as follows:

$$V_S = \{(\mathbf{d}, \mathbf{w}) \in H^1(\mathcal{N}; \mathbb{R}^6) : \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i = 0, i = 1, \dots, n_E\}. \tag{3.7}$$

For a function $(\mathbf{d}, \mathbf{w}) \in V_S$ we consider the following norms on $H^1(\mathcal{N}; \mathbb{R}^3)$:

$$\|\mathbf{d}\|^2_{H^1(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{d}_i\|^2_{H^1(0,l_i;\mathbb{R}^3)}, \quad \|\mathbf{w}\|^2_{H^1(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_{H^1(0,l_i;\mathbb{R}^3)},$$

and the following norms on $L^2(\mathcal{N}; \mathbb{R}^3)$:

$$\|\mathbf{d}\|^2_{L^2(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{d}_i\|^2_{L^2(0,l_i;\mathbb{R}^3)}, \quad \|\mathbf{w}\|^2_{L^2(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_{L^2(0,l_i;\mathbb{R}^3)}.$$

We obtain the weak formulation for a single curved rod model in a standard way. More precisely, by summing up the first equation in (3.6) multiplied by a test function $\boldsymbol{\xi}$ and the second equation in (3.6) multiplied by a test function $\boldsymbol{\zeta}$, integrating over $[0, l]$ (we omit the subscript i in the following calculation), using integration by parts and involving the constitutive relation and the condition of inextensibility and unshearability, we get the weak formulation for a single curved rod model: find $(\mathbf{d}, \mathbf{w})$ such that

$$\begin{aligned} \rho_S A \int_0^l \partial_t^2 \mathbf{d} \cdot \boldsymbol{\xi} + \rho_S \int_0^l M \partial_t^2 \mathbf{w} \cdot \boldsymbol{\zeta} + \int_0^l Q H Q^T \partial_s \mathbf{w} \cdot \partial_s \boldsymbol{\zeta} \\ = \int_0^l \mathbf{f} \cdot \boldsymbol{\xi} + \mathbf{p}(l) \cdot \boldsymbol{\xi}(l) - \mathbf{p}(0) \cdot \boldsymbol{\xi}(0) + \mathbf{q}(l) \cdot \boldsymbol{\zeta}(l) - \mathbf{q}(0) \cdot \boldsymbol{\zeta}(0), \end{aligned} \tag{3.8}$$

for all $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in H^1(0, l) \times H^1(0, l)$ that satisfy the condition of inextensibility and unshearability.

To obtain a weak formulation for the mesh net problem, we sum up the weak formulations for each mesh component (i.e. curved rod). At each vertex, the boundary terms from (3.8) involving $\mathbf{p}$ and $\mathbf{q}$ will sum up to zero due to the dynamic coupling conditions. The weak formulation for the mesh net problem then reads: find $(\mathbf{d}, \mathbf{w}) \in V_S$ such that

$$\begin{aligned} \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t^2 \mathbf{d}_i \cdot \boldsymbol{\xi}_i + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t^2 \mathbf{w}_i \cdot \boldsymbol{\zeta}_i \\ + \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i = \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i \cdot \boldsymbol{\xi}_i, \end{aligned} \tag{3.9}$$

for all test functions $(\boldsymbol{\xi}, \boldsymbol{\zeta}) = ((\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1), \ldots, (\boldsymbol{\xi}_{n_E}, \boldsymbol{\zeta}_{n_E})) \in V_S$.

3.1.4 Coupling between the shell and the elastic mesh

We will be assuming that the elastic mesh is fixed to the shell surface

$$\cup_{i=1}^{n_E} \mathbf{P}_i([0, l_i]) \subset \Gamma = \boldsymbol{\varphi}(\overline{\omega}).$$

Since φ is injective on ω, functions $\boldsymbol{\pi}_i$

$$\boldsymbol{\pi}_i = \boldsymbol{\varphi}^{-1} \circ \mathbf{P}_i : [0, l_i] \to \overline{\omega}, \quad i = 1, \ldots, n_E,$$

are well defined. The reference configuration of the mesh on ω will be denoted by $\omega_S = \cup_{i=1}^{n_E} \boldsymbol{\pi}_i([0, l_i])$ and the reference configuration of the mesh on Γ by $\Gamma_S = \cup_{i=1}^{n_E} \mathbf{P}_i([0, l_i])$.

The elastic mesh and the shell are coupled through the kinematic and dynamic coupling conditions. The kinematic coupling condition states that the displacement of the shell at the point $(z, R\cos\theta, R\sin\theta) \in \Gamma$, that is associated with the point $(z, \theta) \in \omega_S$ via the mapping $\boldsymbol{\varphi}$, is equal to the displacement of the mesh at the point $s_i = \boldsymbol{\pi}_i^{-1}(z, \theta)$, that is associated to the same point $(z, \theta) \in \omega_S$ via the mapping $\boldsymbol{\pi}_i$. For a point $s_i \in [0, l_i]$ such that $\boldsymbol{\pi}_i(s_i) = (z, \theta) \in \omega_S$, the kinematic coupling condition reads:

$$\boldsymbol{\eta}(t, \boldsymbol{\pi}_i(s_i)) = \mathbf{d}_i(t, s_i). \tag{3.10}$$

The dynamic coupling condition describes the balance of forces. The force exerted by the Koiter shell onto the mesh is balanced by the force exerted by the mesh onto the Koiter shell. More precisely, let $J_i = \boldsymbol{\pi}_i([0, l_i])$, and

$$\langle \delta_{J_i}, f \rangle = \int_{J_i} f d\gamma_i, \quad i = 1, \dots, n_E,$$

where $d\gamma_i$ is the curve element associated with the parameterization $\boldsymbol{\pi}_i$. The weak formulation of the shell (3.4) can then be written as:

$$\begin{aligned}
\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi} \rangle &= \sum_{i=1}^{n_E} \langle \delta_{J_i}, \mathbf{f} \cdot \boldsymbol{\psi} R \rangle \\
&= \sum_{i=1}^{n_E} \int_{J_i} \mathbf{f}(z, \theta) \cdot \boldsymbol{\psi}(z, \theta) R d\gamma_i \\
&= \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}(\boldsymbol{\pi}_i(s)) \cdot \boldsymbol{\psi}(\boldsymbol{\pi}_i(s)) \|\boldsymbol{\pi}_i'(s)\| R ds.
\end{aligned}$$

If we denote by $\mathbf{f}_i$ the force exerted by the i-th mesh strut onto the shell, by force balance, the right-hand side has to be equal to $-\sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i(s) \cdot \boldsymbol{\xi}_i(s) ds$. Thus, $\mathbf{f}(\boldsymbol{\pi}_i(s_i)) \|\boldsymbol{\pi}_i'(s_i)\| R = -\mathbf{f}_i(s_i)$, i.e. $\mathbf{f}(\boldsymbol{\pi}_i(s_i)) R = -\dfrac{\mathbf{f}_i(s_i)}{\|\boldsymbol{\pi}_i'(s_i)\|}$, $s_i \in (0, l_i)$. For a point $(z, \theta) = \boldsymbol{\pi}_i(s_i) \in \omega_S$, which came from an $s_i \in (0, l_i)$, the dynamic coupling condition reads: $\mathbf{f} R = -\dfrac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|}$. For an arbitrary point $(z, \theta) \in \omega$, the dynamic coupling condition reads:

$$\mathbf{f} R = -\sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i}. \tag{3.11}$$

Now, the weak formulation for the coupled mesh-shell problem reads:

$$\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi} \rangle = -\sum_{i=1}^{n_E} \langle \delta_{J_i}, \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \boldsymbol{\psi} \rangle, \tag{3.12}$$

for all test functions $\boldsymbol{\psi} \in V_K$. Here $\mathbf{f}_i$'s are defined in (3.9), and the test functions $\boldsymbol{\xi}_i$ are such that $\boldsymbol{\psi} \circ \boldsymbol{\pi}_i = \boldsymbol{\xi}_i$. The coupled mesh-shell weak solution space is given by:

$$V_{KS} = \{(\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_K \times V_S : \boldsymbol{\eta} \circ \boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E} (0, l_i)\},$$

where we denoted $\boldsymbol{\eta} \circ \boldsymbol{\pi} = (\boldsymbol{\eta} \circ \boldsymbol{\pi}_1, \dots, \boldsymbol{\eta} \circ \boldsymbol{\pi}_{n_E})$.

3.1.5 Coupling between the composite structure and the fluid

From now on, by 'composite structure' we denote the Koiter shell coupled with the elastic mesh. The coupling between the fluid and the structure is defined by two sets of boundary conditions satisfied at the lateral boundary $\Gamma^{\boldsymbol{\eta}}(t)$, giving rise to a *nonlinear fluid-structure coupling.* They are the kinematic and dynamic boundary conditions, describing continuity of velocity and balance of contact forces.

- the kinematic condition: $\partial_t \boldsymbol{\eta} = (\mathbf{u} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi}$ on $(0,T) \times \omega$,
- the dynamic condition: $\rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} = -J((\boldsymbol{\sigma} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi})((\mathbf{n}^{\boldsymbol{\eta}} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi})$ on $(0,T) \times \omega$,

where J denotes the Jacobian of the composition of the transformation from Eulerian to Lagrangian coordinates and the transformation from Cartesian to cylindrical coordinates, and $\mathbf{n}^{\boldsymbol{\eta}}$ denotes the outer unit normal on $\Gamma^{\boldsymbol{\eta}}(t)$ at point $\boldsymbol{\phi}^{\boldsymbol{\eta}}(t, \boldsymbol{\varphi}(z,\theta))$. We emphasize that the dynamic coupling condition also reflects the presence of a one-dimensional elastic mesh at the fluid-structure interface.

In summary, we study the following fluid-structure interaction problem:

Problem 1. Find $(\mathbf{u}, p, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ such that

$$\left.\begin{array}{rcl} \rho_F\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &=& \nabla\cdot\boldsymbol{\sigma} \\ \nabla\cdot\mathbf{u} &=& 0 \end{array}\right\} \text{ in } \Omega^{\boldsymbol{\eta}}(t), t \in (0,T), \tag{3.13}$$

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{\eta} &=& (\mathbf{u} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi} \\ \rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} &=& -J((\boldsymbol{\sigma} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi})((\mathbf{n}^{\boldsymbol{\eta}} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi}) \end{array}\right\} \text{on } (0,T)\times\omega, \tag{3.14}$$

$$\left.\begin{array}{rcl} \rho_S A_i \partial_t^2 \mathbf{d}_i &=& \partial_s \mathbf{p}_i + \mathbf{f}_i \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &=& \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i \\ 0 &=& \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i \\ 0 &=& \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i \end{array}\right\} \text{ on } (0,T) \times (0, l_i). \tag{3.15}$$

Problem (3.13)-(3.15) is supplemented with the following set of boundary and initial conditions:

$$\left\{\begin{array}{rcll} p + \frac{\rho_F}{2}|\mathbf{u}|^2 &=& P_{in/out}(t), & \text{on } (0,T)\times\Gamma_{in/out}, \\ \mathbf{u}\times\mathbf{e}_z &=& 0, & \text{on } (0,T)\times\Gamma_{in/out}, \\ \boldsymbol{\eta}(t,0,\theta) &=& \boldsymbol{\eta}(t,L,\theta) = 0, & \text{on } (0,T)\times(0,2\pi), \\ \partial_z \eta_r(t,0,\theta) &=& \partial_z \eta_r(t,L,\theta) = 0, & \text{on } (0,T)\times(0,2\pi), \\ \boldsymbol{\eta}(t,z,0) &=& \boldsymbol{\eta}(t,z,2\pi), & \text{on } (0,T)\times(0,L), \\ \partial_\theta \eta_r(t,z,0) &=& \partial_\theta \eta_r(t,z,2\pi), & \text{on } (0,T)\times(0,L), \end{array}\right. \tag{3.16}$$

$$\begin{aligned} \mathbf{u}(0) = \mathbf{u}_0,\ \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0,\ \partial_t \boldsymbol{\eta}(0) = \mathbf{v}_0, \\ \mathbf{d}_i(0) = \mathbf{d}_{0i},\ \partial_t \mathbf{d}_i(0) = \mathbf{k}_{0i},\ \mathbf{w}_i(0) = \mathbf{w}_{0i},\ \partial_t \mathbf{w}_i(0) = \mathbf{z}_{0i}. \end{aligned} \tag{3.17}$$

This is a nonlinear, moving-boundary problem in 3D, which captures the full, two way fluid-structure interaction coupling. The nonlinearity in the problem is represented by the quadratic term in the fluid equations, the moving boundary, whose position is one of the unknowns of the problem and by the nonlinear coupling between the fluid and structure defined at the moving boundary $\Gamma^{\boldsymbol{\eta}}(t)$.

3.2 The energy of the problem

Problem (3.13)-(3.15) satisfies the following energy inequality:

$$\frac{d}{dt} E(t) + D(t) \leq C(P_{in}(t), P_{out}(t)), \tag{3.18}$$

where $E(t)$ denotes the total energy of the problem (the sum of the kinetic and elastic energy), while $D(t)$ denotes dissipation due to fluid viscosity. $C(P_{in}(t), P_{out}(t))$ is a constant which depends only on the inlet and outlet pressure data, which are both functions of time. More precisely, if we denote by $\|\mathbf{w}\|_m$ and $\|\mathbf{w}\|_r$ the following norms associated with the elastic energy of the mesh:

$$\|\mathbf{w}\|_m^2 := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|_m^2 = \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{w}_i \cdot \mathbf{w}_i,$$

$$\|\mathbf{w}\|_r^2 := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|_r^2 = \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \mathbf{w}_i,$$

and by $\|\boldsymbol{\eta}\|_{L^2(R;\omega)}$ the weighted L^2 norm on ω, with the weight R associated with the geometry of the Koiter shell (Jacobian):

$$\|\boldsymbol{\eta}\|^2_{L^2(R;\omega)} := \int_\omega |\boldsymbol{\eta}|^2 R \, d\omega,$$

then the total energy of the coupled FSI problem is defined by

$$\begin{aligned} E(t) = \frac{\rho_F}{2} \|\mathbf{u}\|^2_{L^2(\Omega^{\boldsymbol{\eta}}(t))} + \frac{\rho_K h}{2} \|\partial_t \boldsymbol{\eta}\|^2_{L^2(R;\omega)} + \frac{\rho_S}{2} \sum_{i=1}^{n_E} A_i \|\partial_t \mathbf{d}_i\|^2_{L^2(0,l_i)} \\ + \frac{\rho_S}{2} \|\partial_t \mathbf{w}\|_m^2 + \frac{1}{2} \langle \mathcal{L} \boldsymbol{\eta}, \boldsymbol{\eta} \rangle + \|\mathbf{w}\|_r^2, \end{aligned}$$

while $D(t)$ is given by

$$D(t) = 2\mu_F \|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega^{\boldsymbol{\eta}}(t))}.$$

Remark The norm $\|\cdot\|_m$ is equivalent to the standard $L^2(\mathcal{N})$ norm.

We derive formal energy inequality (3.18) in a standard way. First, we multipy first equation in (3.1) by $\mathbf{u}$ and integrate over $\Omega^{\boldsymbol{\eta}}(t)$ to obtain:

$$\int_{\Omega^{\boldsymbol{\eta}}(t)} \rho_F \left(\partial_t \mathbf{u} \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u}\right) = \int_{\Omega^{\boldsymbol{\eta}}(t)} (\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}.$$

Having in mind that the velocity of the lateral boundary is equal to $\mathbf{u}|_{\Gamma^{\boldsymbol{\eta}}(t)}$, we use Reynold's transport theorem to deal with the first term on the left-hand side of the previous equation:

$$\begin{aligned}\int_{\Omega^{\boldsymbol{\eta}}(t)} \partial_t \mathbf{u} \cdot \mathbf{u} &= \int_{\Omega^{\boldsymbol{\eta}}(t)} \partial_t \left(\frac{|\mathbf{u}|^2}{2}\right) = \frac{d}{dt}\int_{\Omega^{\boldsymbol{\eta}}(t)} \frac{|\mathbf{u}|^2}{2} - \int_{\partial\Omega^{\boldsymbol{\eta}}(t)} \frac{|\mathbf{u}|^2}{2}\mathbf{u} \cdot \mathbf{n} \\ &= \frac{1}{2}\frac{d}{dt}\int_{\Omega^{\boldsymbol{\eta}}(t)} |\mathbf{u}|^2 - \frac{1}{2}\int_{\Gamma^{\boldsymbol{\eta}}(t)} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n}.\end{aligned}$$

The convective part of the Navier-Stokes equations can be rewritten by using integration by parts and the divergence-free condition:

$$\begin{aligned}\int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} &= \frac{1}{2}\int_{\Omega^{\boldsymbol{\eta}}(t)} \nabla \cdot (|\mathbf{u}|^2 \mathbf{u}) - \int_{\Omega^{\boldsymbol{\eta}}(t)} (\nabla \cdot \mathbf{u})|\mathbf{u}|^2 \\ &= \frac{1}{2}\int_{\partial\Omega^{\boldsymbol{\eta}}(t)} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n} \\ &= \frac{1}{2}\int_{\Gamma^{\boldsymbol{\eta}}(t)} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n} - \frac{1}{2}\int_{\Gamma_{in}} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{e}_z + \frac{1}{2}\int_{\Gamma_{out}} |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{e}_z.\end{aligned}$$

These two terms added together give:

$$\begin{aligned}\int_{\Omega^{\boldsymbol{\eta}}(t)} \partial_t \mathbf{u} \cdot \mathbf{u} + \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u} = \frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega^{\boldsymbol{\eta}}(t))} - \frac{1}{2}\int_{\Gamma_{in}} |\mathbf{u}|^2 u_z \\ + \frac{1}{2}\int_{\Gamma_{out}} |\mathbf{u}|^2 u_z.\end{aligned} \tag{3.19}$$

Now we use integration by parts to deal with the right-hand side in the Navier-Stokes equations:

$$\begin{aligned}\int_{\Omega^{\boldsymbol{\eta}}(t)} (\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} &= \int_{\Omega^{\boldsymbol{\eta}}(t)} \nabla \cdot (\boldsymbol{\sigma}\mathbf{u}) - \int_{\Omega^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma} : \nabla \mathbf{u} \\ &= \int_{\partial\Omega^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{u} \cdot \mathbf{n} + \int_{\Omega^{\boldsymbol{\eta}}(t)} p\mathbf{I} : \nabla \mathbf{u} - 2\mu_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{D}(\mathbf{u}) : \nabla \mathbf{u} \\ &= \int_{\partial\Omega^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{u} \cdot \mathbf{n} + \int_{\Omega^{\boldsymbol{\eta}}(t)} p\nabla \cdot \mathbf{u} - 2\mu_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) \\ &= \int_{\partial\Omega^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{u} \cdot \mathbf{n} - 2\mu_F \int_{\Omega^{\boldsymbol{\eta}}(t)} |\mathbf{D}(\mathbf{u})|^2.\end{aligned}$$

To deal with the boundary integral over $\partial\Omega^{\boldsymbol{\eta}}(t)$ we first notice that on $\Gamma_{in/out}$ the boundary condition $\mathbf{u} \times \mathbf{e}_z = 0$ implies $u_x = u_y = 0$. Using divergence-free condition, we also obtain $\partial_z u_z = 0$. These two facts combined together imply that $\mathbf{D}(\mathbf{u})\mathbf{u} \cdot \mathbf{n} = 0$ on $\Gamma_{in/out}$. Finally, since the normal on $\Gamma_{in/out}$ is equal to $\mathbf{n} = (\mp 1, 0, 0)$, we get:

$$\begin{aligned}\int_{\partial\Omega^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{u}\cdot\mathbf{n} &= \int_{\Gamma^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{u}\cdot\mathbf{n} + \int_{\Gamma_{in/out}} \left(-p\mathbf{I} + 2\mu_F\mathbf{D}(\mathbf{u})\right)\mathbf{u}\cdot\mathbf{n} \\ &= \int_{\Gamma^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{u}\cdot\mathbf{n} + \int_{\Gamma_{in}} pu_z - \int_{\Gamma_{out}} pu_z.\end{aligned} \tag{3.20}$$

What is left is to calculate the boundary integral over $\Gamma^{\boldsymbol{\eta}}(t)$. To do that, we first multiply the Koiter shell equation (3.5) by $\partial_t\boldsymbol{\eta}$ and integrate over ω :

$$\begin{aligned}\int_\omega \mathbf{f}\cdot\partial_t\boldsymbol{\eta}R &= \rho_K h\int_\omega \partial_t^2\boldsymbol{\eta}\cdot\partial_t\boldsymbol{\eta}R + \langle\boldsymbol{\mathcal{L}}\boldsymbol{\eta},\partial_t\boldsymbol{\eta}\rangle \\ &= \frac{\rho_K h}{2}\frac{d}{dt}\int_\omega \partial_t\boldsymbol{\eta}\cdot\partial_t\boldsymbol{\eta}R + \frac{1}{2}\frac{d}{dt}\langle\boldsymbol{\mathcal{L}}\boldsymbol{\eta},\boldsymbol{\eta}\rangle \\ &= \frac{\rho_K h}{2}\frac{d}{dt}\|\partial_t\boldsymbol{\eta}\|^2_{L^2(R;\omega)} + \frac{1}{2}\frac{d}{dt}\langle\boldsymbol{\mathcal{L}}\boldsymbol{\eta},\boldsymbol{\eta}\rangle.\end{aligned}$$

Next, we want to use $(\partial_t\mathbf{d}_i, \partial_t\mathbf{w}_i), i = 1,\dots,n_E$, as test functions in the weak formulation for the mesh problem (3.9). Before doing so, we need to check if these test functions are *admissible*, i.e. if they satisfy the condition of inextensibility and unshearability $\partial_s\mathbf{d}_i + \mathbf{t}_i\times\mathbf{w}_i = 0, i = 1,\dots,n_E$. We differentiate this condition with respect to t and use the fact that $\mathbf{t}_i$ do not depend on t, to obtain

$$\partial_s(\partial_t\mathbf{d}_i) + \mathbf{t}_i\times(\partial_t\mathbf{w}_i) = 0, \quad i = 1,\dots,n_E,$$

which implies that, indeed, $(\partial_t\mathbf{d}_i, \partial_t\mathbf{w}_i) \in V_S,\ i = 1,\dots,n_E$.

By using $(\partial_t\mathbf{d}, \partial_t\mathbf{w}) = ((\partial_t\mathbf{d}_1, \partial_t\mathbf{w}_1),\dots,(\partial_t\mathbf{d}_{n_E}, \partial_t\mathbf{w}_{n_E}))$ as a test function in the weak formulation for the mesh problem, we obtain:

$$\begin{aligned}\sum_{i=1}^{n_E}\int_0^{l_i}\mathbf{f}_i\cdot\partial_t\mathbf{d}_i &= \rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\partial_t^2\mathbf{d}_i\cdot\partial_t\mathbf{d}_i + \rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\partial_t^2\mathbf{w}_i\cdot\partial_t\mathbf{w}_i \\ &\quad + \sum_{i=1}^{n_E}\int_0^{l_i}Q_iH_iQ_i^T\partial_s\mathbf{w}_i\cdot\partial_s\partial_t\mathbf{w}_i \\ &= \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}A_i\|\partial_t\mathbf{d}_i\|^2_{L^2(0,l_i)} + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}\|\partial_t\mathbf{w}_i\|^2_m \\ &\quad + \frac{d}{dt}\sum_{i=1}^{n_E}\|\mathbf{w}_i\|^2_r.\end{aligned}$$

By enforcing the kinematic and the dynamic boundary conditions on ω, we obtain:

$$\begin{aligned} -\int_{\Gamma^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u} &= -\int_{\omega} J((\boldsymbol{\sigma}\circ\boldsymbol{\phi}^{\boldsymbol{\eta}})|_{\Gamma}\circ\boldsymbol{\varphi})((\mathbf{n}\circ\boldsymbol{\phi}^{\boldsymbol{\eta}})|_{\Gamma}\circ\boldsymbol{\varphi})\cdot((\mathbf{u}\circ\phi^{\boldsymbol{\eta}})|_{\Gamma}\circ\boldsymbol{\varphi}) \\ &= \int_{\omega}\mathbf{f}\cdot\partial_t\boldsymbol{\eta}R + \sum_{i=1}^{n_E}\int_{J_i}\frac{\mathbf{f}_i\circ\boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i'\circ\boldsymbol{\pi}_i^{-1}\|}\delta_{J_i}\cdot\partial_t\boldsymbol{\eta} \\ &= \int_{\omega}\mathbf{f}\cdot\partial_t\boldsymbol{\eta}R + \sum_{i=1}^{n_E}\int_0^{l_i}\mathbf{f}_i\cdot\partial_t\boldsymbol{\eta}\circ\boldsymbol{\pi}_i \\ &= \int_{\omega}\mathbf{f}\cdot\partial_t\boldsymbol{\eta}R + \sum_{i=1}^{n_E}\int_0^{l_i}\mathbf{f}_i\cdot\partial_t\mathbf{d}_i. \end{aligned} \tag{3.21}$$

By inserting the previous calculation for $\mathbf{f}$ and $\mathbf{f}_i$ from the shell and mesh problems into (3.21), we get:

$$\begin{aligned} -\int_{\Gamma^{\boldsymbol{\eta}}(t)} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u} = \frac{\rho_K h}{2}\frac{d}{dt}\|\partial_t\boldsymbol{\eta}\|^2_{L^2(R;\omega)} + \frac{1}{2}\frac{d}{dt}\langle\mathcal{L}\boldsymbol{\eta},\boldsymbol{\eta}\rangle \\ + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}A_i\|\partial_t\mathbf{d}_i\|^2_{L^2(0,l_i)} + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}\|\partial_t\mathbf{w}_i\|^2_m \\ + \frac{d}{dt}\sum_{i=1}^{n_E}\|\mathbf{w}_i\|^2_r. \end{aligned} \tag{3.22}$$

Finally, by combining (3.22) with (3.20), and by adding the remaining contributions to the energy of the FSI problem calculated in equations (3.20) and (3.19), one obtains the following energy equality:

$$\begin{aligned} \frac{\rho_F}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega^{\boldsymbol{\eta}}(t))} + 2\mu_F\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega^{\boldsymbol{\eta}}(t))} + \frac{\rho_K h}{2}\frac{d}{dt}\|\partial_t\boldsymbol{\eta}\|^2_{L^2(R;\omega)} \\ + \frac{1}{2}\frac{d}{dt}\langle\mathcal{L}\boldsymbol{\eta},\boldsymbol{\eta}\rangle + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}A_i\|\partial_t\mathbf{d}_i\|^2_{L^2(0,l_i)} + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}\|\partial_t\mathbf{w}_i\|^2_m \\ + \frac{d}{dt}\sum_{i=1}^{n_E}\|\mathbf{w}_i\|^2_r = \int_{\Gamma_{in}}\left(p+\frac{\rho_F}{2}|\mathbf{u}|^2\right)u_z - \int_{\Gamma_{out}}\left(p+\frac{\rho_F}{2}|\mathbf{u}|^2\right)u_z. \end{aligned} \tag{3.23}$$

What is left is to bound the right-hand side which is equal to

$$\int_{\Gamma_{in}} P_{in}(t)u_z - \int_{\Gamma_{out}} P_{out}(t)u_z.$$

Using the trace theorem, Korn's inequality and Cauchy inequality (with ε), one can estimate:

$$\begin{aligned} \left|\int_{\Gamma_{in/out}} P_{in/out}(t)u_z\right| &\le C|P_{in/out}|\|\mathbf{u}\|_{H^1(\Omega^{\boldsymbol{\eta}}(t))} \le C|P_{in/out}|\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega^{\boldsymbol{\eta}}(t))} \\ &\le \frac{C}{2\varepsilon}|P_{in/out}|^2 + \frac{C\varepsilon}{2}\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega^{\boldsymbol{\eta}}(t))}. \end{aligned}$$

We note that the fluid velocity $\mathbf{u}$ satisfies the conditions for Korn's inequality (Theorem 6.3-

4 in [29]). Namely, the boundary condition $\mathbf{u} \times \mathbf{e}_z = 0$ on $\Gamma_{in/out}$ gives us $u_x = u_y = 0$ and $\partial_z u_z = 0$. Using the kinematic coupling condition $u_z = \partial_t \eta_z$ (on ω), we obtain that $u_z = 0$, so $\mathbf{u} = 0$ on $\Gamma_{in/out}$. Finally, by choosing ε such that $\frac{C\varepsilon}{2} \leq \mu_F$, we get the energy inequality (3.18).

3.3 Existence of weak solutions

3.3.1 Function spaces

To define a weak solution to Problem 1 we introduce the necessary function spaces. For the fluid velocity we will need the classical function space:

$$V_F(t) = \{\mathbf{u} \in H^1(\Omega^{\boldsymbol{\eta}}(t)) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \times \mathbf{e}_z = 0 \text{ on } \Gamma_{in/out}\}. \tag{3.24}$$

Motivated by the energy inequality (3.18), we also define the following evolution spaces associated with the fluid problem, the Koiter shell problem, the elastic mesh problem, and the coupled mesh-shell problem:

- $V_F(0,T) = L^\infty(0,T; L^2(\Omega^{\boldsymbol{\eta}}(t))) \cap L^2(0,T; V_F(t))$,
  where $V_F(t)$ is defined by (3.24),
- $V_K(0,T) = W^{1,\infty}(0,T; L^2(R;\omega)) \cap L^\infty(0,T; V_K)$,
  where V_K is defined by (3.3),
- $V_S(0,T) = W^{1,\infty}(0,T; L^2(\mathcal{N})) \cap L^\infty(0,T; V_S)$,
  where V_S is defined by (3.7),
- $V_{KS}(0,T) = \{(\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_K(0,T) \times V_S(0,T) : \boldsymbol{\eta} \circ \boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E}(0,l_i)\}$.

The solution space for the coupled fluid-mesh-shell interaction problem must involve the kinematic coupling condition, which is, thus, enforced in a strong way:

$$\mathcal{V}(0,T) = \{(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_F(0,T) \times V_{KS}(0,T) : (\mathbf{u} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi} = \partial_t \boldsymbol{\eta} \text{ on } \omega\}.$$

The corresponding test space will be denoted by:

$$\mathcal{Q}(0,T) = \{(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in C_c^1([0,T); V_F \times V_{KS}) : (\boldsymbol{\upsilon} \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi} = \boldsymbol{\psi} \text{ on } \omega\}.$$

3.3.2 Definition of a weak solution

We are now in position to define weak solutions of our moving-boundary, fluid-mesh-shell interaction problem, defined on the moving domain $\Omega^{\boldsymbol{\eta}}(t)$.

Definition 3.1 We say that $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in \mathcal{V}(0,T)$ is a weak solution of Problem 1 if for all test functions $(\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in \mathcal{Q}(0,T)$ the following equality holds:

$$
\begin{aligned}
&\rho_F \left(-\int_0^T \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{u} \cdot \partial_t \boldsymbol{v} + \int_0^T b(t, \mathbf{u}, \mathbf{u}, \boldsymbol{v}) - \frac{1}{2} \int_0^T \int_{\Gamma^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \boldsymbol{v})(\mathbf{u} \cdot \mathbf{n}^{\boldsymbol{\eta}}) \right) \\
&+ 2\mu_F \int_0^T \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{v}) - \rho_K h \int_0^T \int_\omega \partial_t \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\psi} R + \int_0^T a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) \\
&- \rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} \partial_t \mathbf{d}_i \cdot \partial_t \boldsymbol{\xi}_i - \rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i \partial_t \mathbf{w}_i \cdot \partial_t \boldsymbol{\zeta}_i \\
&+ \int_0^T a_S(\mathbf{w}, \boldsymbol{\zeta}) = \int_0^T \langle F(t), \boldsymbol{v} \rangle_{\Gamma_{in/out}} + \rho_F \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{v}(0) + \rho_K h \int_\omega \partial_t \boldsymbol{\eta}_0 \cdot \boldsymbol{\psi}(0) R \\
&+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t \mathbf{d}_{0i} \cdot \boldsymbol{\xi}_i(0) + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t \mathbf{w}_{0i} \cdot \boldsymbol{\zeta}_i(0),
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
b(t, \mathbf{u}, \mathbf{u}, \boldsymbol{v}) &= \frac{1}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{v} - \frac{1}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \boldsymbol{v} \cdot \mathbf{u}, \\
a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) &= \langle \mathcal{L} \boldsymbol{\eta}, \boldsymbol{\psi} \rangle, \\
a_S(\mathbf{w}, \boldsymbol{\zeta}) &= \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i,
\end{aligned}
$$

and

$$
\langle F(t), \boldsymbol{v} \rangle_{\Gamma_{in/out}} = P_{in}(t) \int_{\Gamma_{in}} v_z - P_{out}(t) \int_{\Gamma_{out}} v_z .
$$

In deriving the weak formulation, we used integration by parts in a classical way. Here we present the transformation of the inertial and convective term of the fluid part:

$$
\begin{aligned}
\rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \partial_t \mathbf{u} \cdot \boldsymbol{v} &= \rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \partial_t (\mathbf{u} \cdot \boldsymbol{v}) - \rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{u} \cdot \partial_t \boldsymbol{v} \\
&= \rho_F \frac{d}{dt} \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{u} \cdot \boldsymbol{v} - \rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \boldsymbol{v})(\mathbf{u} \cdot \mathbf{n}) - \rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{u} \cdot \partial_t \boldsymbol{v} \\
&= -\rho_F \mathbf{u}_0 \cdot \boldsymbol{v}(0) - \rho_F \int_{\Gamma^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \boldsymbol{v})(\mathbf{u} \cdot \mathbf{n}) - \rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} \mathbf{u} \cdot \partial_t \boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
\rho_F \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{v} &= \frac{\rho_F}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{v} + \frac{\rho_F}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{v} \\
&= \frac{\rho_F}{2} \int_{\partial\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \boldsymbol{v})(\mathbf{u} \cdot \mathbf{n}) - \frac{\rho_F}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\nabla \cdot \mathbf{u}) \mathbf{u} \cdot \boldsymbol{v} \\
&\quad - \frac{\rho_F}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \boldsymbol{v} \cdot \mathbf{u} + \frac{\rho_F}{2} \int_{\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{v} \\
&= \rho_F b(t, \mathbf{u}, \mathbf{u}, \boldsymbol{v}) + \frac{\rho_F}{2} \int_{\partial\Omega^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \boldsymbol{v})(\mathbf{u} \cdot \mathbf{n}) \\
&= \rho_F b(t, \mathbf{u}, \mathbf{u}, \boldsymbol{v}) + \frac{\rho_F}{2} \int_{\Gamma^{\boldsymbol{\eta}}(t)} (\mathbf{u} \cdot \boldsymbol{v})(\mathbf{u} \cdot \mathbf{n}) \\
&\quad - \frac{\rho_F}{2} \int_{\Gamma_{in}} |\mathbf{u}|^2 v_z + \frac{\rho_F}{2} \int_{\Gamma_{out}} |\mathbf{u}|^2 v_z .
\end{aligned}
$$

3.4 The fluid domain boundary reparameterization

Due to the fact that the structure/shell is moving in all three spatial direction, i.e. all three displacements are non-negligible, the fluid domain boundary may degenerate in such a way that it ceases to be a subgraph of a function. Let us recall that the lateral boundary of the fluid domain (which coincides with the fluid-structure interface) is given by:

$$\Gamma^{\boldsymbol{\eta}}(t) = \{(z + \eta_z(t,z,\theta), R + \eta_r(t,z,\theta), \theta + \eta_\theta(t,z,\theta)) : z \in (0,L), \theta \in (0,2\pi)\}.$$

In order to avoid such degeneracy, we introduce an additional assumption that there exists a time $T > 0$ such that for every $t \leq T, \Gamma^{\boldsymbol{\eta}}(t)$ remains a subgraph of a function. Under this assumption, we reparameterize the lateral boundary of the fluid domain in such a way that the radial displacement becomes the only unknown. More precisely, we introduce new variables

$$\begin{aligned} \tilde{z} &= z + \eta_z(t,z,\theta), \\ \tilde{\eta}(t,\tilde{z},\tilde{\theta}) &= \eta_r(t,z,\theta), \\ \tilde{\theta} &= \theta + \eta_\theta(t,z,\theta), \end{aligned} \tag{3.26}$$

and define the reparameterized lateral boundary of the fluid domain by

$$\Gamma^{\tilde{\boldsymbol{\eta}}}(t) = \{(\tilde{z}, R + \tilde{\eta}(t,\tilde{z},\tilde{\theta}), \tilde{\theta}) : \tilde{z} \in (0,L), \tilde{\theta} \in (0,2\pi)\}. \tag{3.27}$$

It is easy to check that the displacement $\tilde{\eta}$ satisfies the following:

$$\tilde{\eta}(t,\tilde{z},\tilde{\theta}) = \eta_r \circ (\mathrm{id}_z + \eta_z, \mathrm{id}_\theta + \eta_\theta)^{-1}(t,\tilde{z},\tilde{\theta}),$$

where id_z and id_θ are projections of the identity to the second and third variable.

Now, the shell displacement is given by the function $\tilde{\boldsymbol{\eta}} = \tilde{\eta}\mathbf{e}_r$, where $\mathbf{e}_r = \mathbf{e}_r(\theta) = (0, \cos\theta, \sin\theta)$ is the unit vector in the radial direction. We need to check under which conditions is this reparameterization well-defined. For the simplicity of notation, we leave out the time variable and define auxiliary function $\mathbf{g} : \omega \to \mathbb{R}^2$,

$$\mathbf{g}(z,\theta) = (z + \eta_z(z,\theta), \theta + \eta_\theta(z,\theta)). \tag{3.28}$$

Our goal is to prove that $\mathbf{g}$ is an injective function from ω to its image. For now, we leave this as an assumption which will be justified in further sections.

3.4.1 The operator splitting scheme

Our goal is to prove the existence of a weak solution to the coupled FSI problem. The strategy is the following: we will approximate the coupled FSI problem using time-discretization via operator splitting, and then prove that solutions to the approximate problems converge to a weak solution of the continuous problem, as the time-discretization step tends to zero.

We use the Lie splitting scheme which can be summarized as follows. Let $N \in \mathbb{N}, \Delta t = T/N$ and $t_n = n\Delta t, n = 0, \ldots, N-1$ and consider the following initial-value problem:

$$\begin{aligned} \frac{d\phi}{dt} + A\phi = 0 \quad \text{in} \quad (0,T), \\ \phi(0) = \phi_0, \end{aligned}$$

where A is an operator defined on a Hilbert space, and A can be written as $A = A_1 + A_2$. Set $\phi^0 = \phi_0$ and for $n = 0, 1, \ldots, N-1, i = 1, 2$, compute $\phi^{n+\frac{i}{2}}$ by solving

$$\begin{aligned} \frac{d\phi_i}{dt} + A_i\phi_i = 0 \quad \text{in} \quad (t_n, t_{n+1}), \\ \phi_i(t_n) = \phi^{n+\frac{i-1}{2}}, \end{aligned}$$

and then set $\phi^{n+\frac{i}{2}} = \phi_i(t_{n+1})$.

To perform the time-discretization via operator splitting, we need to rewrite our FSI problem as a first order system in time. This will be done by replacing the second-order time derivative of $\boldsymbol{\eta}$, with the first-order time derivative of the Koiter shell velocity $\mathbf{v} = \partial_t\boldsymbol{\eta}$, by replacing the second-order time derivative of $\mathbf{d}$ by the first-order time derivative of the mesh velocity $\mathbf{k} = \partial_t\mathbf{d}$, and by replacing the second-order time derivative of $\mathbf{w}$ by the first-order time derivative of the rotation velocity $\mathbf{z} = \partial_t\mathbf{w}$.

We apply this approach to split Problem 1 into a fluid and a structure subproblem, and then

1. solve the structure subproblem on (t_n, t_{n+1}) using for the initial data the solution of the fluid subproblem from the previous time step, and then

2. solve the fluid subproblem on (t_n, t_{n+1}) using for the initial data the solution of the just calculated structure subproblem.

In the structure subproblem, we solve an elastodynamics problem for the location of the deformable boundary by involving only the elastic energy of the structure. The motion of the structure is driven by the initial velocity obtained from the trace of the fluid velocity in the previous time step. The fluid velocity $\mathbf{u}$ remains unchanged in this step.

In the fluid subproblem, we solve the Navier-Stokes equations with a ”Robin-type” boundary condition on ω/Γ which involves the shell inertia and the trace of the fluid normal stress. Since the fluid and the elastic mesh ”feel” each other only through the motion of the shell, we exclude the mesh from the fluid subproblem. Namely, since we are working with weak solutions of Leray type, the fluid velocity has no trace on the mesh domain since it is a one-dimensional set. Thus, in this step, the structure displacement, the velocity of the mesh displacement and the velocity of infinitesimal rotation of cross sections remain unchanged.

The inclusion of the shell inertia into the fluid subproblem guarantees energy balance at the time-discrete level, thereby avoiding stability problems due to the so called added mass effect. Here we emphasize that there is no added mass effect associated with the mesh since the fluid velocity does not have the trace defined on the $1D$ set describing the mesh, and therefore it is enough to include only the shell inertia into the fluid subproblem.

3.4.2 Arbitrary Lagrangian-Eulerian mapping

Before we apply the Lie splitting method to our fluid-mesh-shell interaction problem, we need to overcome the difficulties that arise due to the motion of the fluid domain boundary. More precisely, at each time step $t_n = n\Delta t,\ n = 0, \dots, N-1$, we will have a different domain

$$\Omega^n = \boldsymbol{\phi}^n(\Omega) := \boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}}(n\Delta t, \Omega) \tag{3.29}$$

on which we have to solve both the structure and the fluid subproblem. It is convenient to have a mapping between the physical fluid domain $\Omega^{\tilde{\boldsymbol{\eta}}}(t)$ and the discrete fluid domain Ω^{n+1}. We define the corresponding Arbitrary Lagrangian-Eulerian mapping in the following way:

$$\mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t) : \Omega^{n+1} \to \Omega^{\tilde{\boldsymbol{\eta}}}(t), \quad \mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t) = \boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}}(t, \cdot) \circ \boldsymbol{\phi}^{n+1}(\cdot)^{-1}.$$

Since we reparameterized the fluid domain boundary, we can construct the ALE mapping $\mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t)$ by the following explicit formula:

$$\mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t) : \Omega^{n+1} \to \Omega^{\tilde{\boldsymbol{\eta}}}(t), \quad \mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t) \begin{pmatrix} \tilde{z} \\ \tilde{r} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} \tilde{z} \\ \frac{R+\tilde{\eta}(t,\tilde{z},\tilde{\theta})}{R+\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})}\tilde{r} \\ \tilde{\theta} \end{pmatrix}, \tag{3.30}$$

where $(\tilde{z}, \tilde{r}, \tilde{\theta})$ denote the cylindrical coordinates in the discrete physical domain Ω^{n+1}. Due to the fact that we are working with the Navier-Stokes equations written in Cartesian coordinates, it is useful to write an explicit form of the ALE mapping $\mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t)$ in the

Cartesian coordinates as well:

$$\mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t)\begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} z \\ \frac{R+\tilde{\eta}(t,\tilde{z},\tilde{\theta})}{R+\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})}x \\ \frac{R+\tilde{\eta}(t,\tilde{z},\tilde{\theta})}{R+\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})}y \end{pmatrix}.$$

Mapping $\mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t)$ is a bijection, and its Jacobian $S^{\tilde{\boldsymbol{\eta}}}$ and the ALE domain velocity $\mathbf{s}^{\tilde{\boldsymbol{\eta}}}$ are respectively given by:

$$\begin{aligned} S^{\tilde{\boldsymbol{\eta}}}(t) &= |\det \nabla \mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t)| = \left| \frac{R+\tilde{\eta}(t,\tilde{z},\tilde{\theta})}{R+\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})} \right|^2, \\ \mathbf{s}^{\tilde{\boldsymbol{\eta}}} &= \partial_t \mathbf{A}^{\tilde{\boldsymbol{\eta}}}(t) = \frac{\partial_t \tilde{\eta}(t,\tilde{z},\tilde{\theta})}{R+\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})}\tilde{r}\mathbf{e}_r. \end{aligned} \tag{3.31}$$

We define the ALE derivative as a time derivative evaluated on the domain Ω^{n+1} :

$$\partial_t \mathbf{u}|_{\Omega^{n+1}} = \partial_t \mathbf{u} + (\mathbf{s}^{\tilde{\boldsymbol{\eta}}} \cdot \nabla)\mathbf{u}.$$

Using the ALE mapping, we can rewrite the Navier-Stokes equations in the ALE formulation as follows:

$$\partial_t \mathbf{u}|_{\Omega^{n+1}} + ((\mathbf{u} - \mathbf{s}^{\tilde{\boldsymbol{\eta}}}) \cdot \nabla)\mathbf{u} = \nabla \cdot \boldsymbol{\sigma}.$$

3.5 Approximate solutions

We use the Lie operator splitting scheme and semi-discretization to define a sequence of approximate solutions to Problem 1. More precisely, we discretize in time each of the subproblems (defined in the previous section) using Backward Euler scheme. Let $\Delta t = T/N$ be the time-discretization parameter so that the time interval $(0,T)$ is subdivided into N subintervals of width Δt. We define the vector of unknown approximate solutions

$$\mathbf{X}_N^{n+i/2} = (\mathbf{u}_N^{n+i/2}, \boldsymbol{\eta}_N^{n+i/2}, \mathbf{v}_N^{n+i/2}, \mathbf{d}_N^{n+i/2}, \mathbf{w}_N^{n+i/2}, \mathbf{k}_N^{n+i/2}, \mathbf{z}_N^{n+1/2}),$$

$n = 0, 1, \ldots, N-1, i = 1, 2$, where $i = 1, 2$ denotes the solution of the structure and the fluid subproblem respectively. The semi-discretization of the problem will be performed in such a way that the discrete version of the energy inequality (3.18) is preserved at every time step. We define the semi-discrete versions of the kinetic and elastic energy, and of dissipation, by the following:

$$
\begin{aligned}
E_N^{n+i/2} = \rho_F \int_{\Omega^{n+1}} |\mathbf{u}_N^{n+1}|^2 + \rho_K h \int_\omega |\mathbf{v}_N^{n+i/2}|^2 R + a_K(\boldsymbol{\eta}_N^{n+i/2}, \boldsymbol{\eta}_N^{n+i/2}) \\
+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} |(\mathbf{k}_N^{n+i/2})_i|^2 + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i |(\mathbf{z}_N^{n+i/2})_i|^2 \\
+ a_S(\mathbf{w}_N^{n+i/2}, \mathbf{w}_N^{n+i/2}), \quad i = 1, 2,
\end{aligned}
\tag{3.32}
$$

$$
D_N^{n+1} = 2\Delta t \mu_F \int_{\Omega^{n+1}} |\mathbf{D}(\mathbf{u}_N^{n+1})|^2, \quad n = 0, 1, \dots, N-1. \tag{3.33}
$$

Throughout the rest of this section, we fix the time step Δt, i.e. we keep $N \in \mathbb{N}$ fixed, and study the semi-discretized subproblems defined by the Lie splitting. To simplify notation, we omit the subscript N and write $(\mathbf{u}^{n+i/2}, \boldsymbol{\eta}^{n+i/2}, \mathbf{v}^{n+i/2}, \mathbf{d}^{n+i/2}, \mathbf{w}^{n+i/2}, \mathbf{k}^{n+i/2}, \mathbf{z}^{n+1/2})$, instead of $(\mathbf{u}_N^{n+i/2}, \boldsymbol{\eta}_N^{n+i/2}, \mathbf{v}_N^{n+i/2}, \mathbf{d}_N^{n+i/2}, \mathbf{w}_N^{n+i/2}, \mathbf{k}_N^{n+i/2}, \mathbf{z}_N^{n+1/2})$.

3.5.1 The semi-discretized structure subproblem

We write a semi-discrete version of the composite structure subproblem defined by the Lie splitting. In this step $\mathbf{u}$ does not change, so

$$
\mathbf{u}^{n+1/2} = \mathbf{u}^n.
$$

Furthermore, we define $(\boldsymbol{\eta}^{n+1/2}, \mathbf{v}^{n+1/2}, \mathbf{d}^{n+1/2}, \mathbf{w}^{n+1/2}, \mathbf{k}^{n+1/2}, \mathbf{z}^{n+1/2}) \in W_S$ as the solution of the following problem, written in weak form:

$$
\begin{aligned}
&\rho_K h \int_\omega \frac{\mathbf{v}^{n+1/2} - \mathbf{v}^n}{\Delta t} \cdot \boldsymbol{\psi} R + a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\psi}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{\mathbf{k}_i^{n+1/2} - \mathbf{k}_i^n}{\Delta t} \cdot \boldsymbol{\xi}_i \\
&\qquad + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{\mathbf{z}_i^{n+1/2} - \mathbf{z}_i^n}{\Delta t} \cdot \boldsymbol{\zeta}_i + a_S(\mathbf{w}^{n+1/2}, \boldsymbol{\zeta}) = 0, \\
&\int_\omega \frac{\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n}{\Delta t} \cdot \boldsymbol{\psi} R = \int_\omega \mathbf{v}^{n+1/2} \cdot \boldsymbol{\psi} R, \\
&\left.\begin{aligned}
\int_0^{l_i} \frac{\mathbf{d}_i^{n+1/2} - \mathbf{d}_i^n}{\Delta t} \cdot \boldsymbol{\xi}_i &= \int_0^{l_i} \mathbf{k}_i^{n+1/2} \cdot \boldsymbol{\xi}_i, \\
\int_0^{l_i} \frac{\mathbf{w}_i^{n+1/2} - \mathbf{w}_i^n}{\Delta t} \cdot \boldsymbol{\zeta}_i &= \int_0^{l_i} \mathbf{z}_i^{n+1/2} \cdot \boldsymbol{\zeta}_i,
\end{aligned}\right\} \ i = 1, \dots, n_E,
\end{aligned}
\tag{3.34}
$$

for all test functions $(\boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in V_{KS}$, where the solution space is defined by:

$$
\begin{aligned}
W_S := \{(\boldsymbol{\eta}, \mathbf{v}, \mathbf{d}, \mathbf{w}, \mathbf{k}, \mathbf{z}) \in V_K \times H^2(\omega) \times V_S \times H^1(\mathcal{N}) \times H^1(\mathcal{N}) : \\
\boldsymbol{\eta} \circ \boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E} (0, l_i)\}.
\end{aligned}
\tag{3.35}
$$

Proposition 3.2 For each fixed $\Delta t > 0$, the structure subproblem has a unique solution $(\boldsymbol{\eta}^{n+1/2}, \mathbf{v}^{n+1/2}, \mathbf{d}^{n+1/2}, \mathbf{w}^{n+1/2}, \mathbf{k}^{n+1/2}, \mathbf{z}^{n+1/2}) \in W_S$.

Proof. The proof is a direct consequence of the Lax-Milgram lemma. To show this, we rewrite the first equation in (3.34) by replacing $\mathbf{v}^{n+1/2}$ by $\dfrac{\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n}{\Delta t}$, $\mathbf{k}_i^{n+1/2}$ by $\dfrac{\mathbf{d}_i^{n+1/2} - \mathbf{d}_i^n}{\Delta t}$ and $\mathbf{z}_i^{n+1/2}$ by $\dfrac{\mathbf{w}_i^{n+1/2} - \mathbf{w}_i^n}{\Delta t}$, for $i = 1, \dots, n_E$. We get

$$
\begin{aligned}
&\rho_K h \int_\omega \frac{\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n}{(\Delta t)^2} \cdot \boldsymbol{\psi} R + a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\psi}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{\mathbf{d}_i^{n+1/2} - \mathbf{d}_i^n}{(\Delta t)^2} \cdot \boldsymbol{\xi}_i \\
&+ \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{\mathbf{w}_i^{n+1/2} - \mathbf{w}_i^n}{(\Delta t)^2} \cdot \boldsymbol{\zeta}_i + a_S(\mathbf{w}^{n+1/2}, \boldsymbol{\zeta}) \\
&= \rho_K h \int_\omega \frac{\mathbf{v}^n}{\Delta t} \cdot \boldsymbol{\psi} R + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{\mathbf{k}_i^n}{\Delta t} \cdot \boldsymbol{\xi}_i + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{\mathbf{z}_i^n}{\Delta t} \cdot \boldsymbol{\zeta}_i.
\end{aligned}
$$

We multiply this equality by $(\Delta t)^2$ and move all the terms from the $n-$th step to the right-hand side to obtain:

$$
\begin{aligned}
&\rho_K h \int_\omega \boldsymbol{\eta}^{n+1/2} \cdot \boldsymbol{\psi} R + (\Delta t)^2 a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\psi}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{d}_i^{n+1/2} \cdot \boldsymbol{\xi}_i \\
&+ \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{w}_i^{n+1/2} \cdot \boldsymbol{\zeta}_i + (\Delta t)^2 a_S(\mathbf{w}^{n+1/2}, \boldsymbol{\zeta}) = \rho_K h \int_\omega \Big(\boldsymbol{\eta}^n + \Delta t \mathbf{v}^n\Big) \cdot \boldsymbol{\psi} R \\
&+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \Big(\mathbf{d}_i^n + \Delta t \mathbf{k}_i^n\Big) \cdot \boldsymbol{\xi}_i + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \Big(\mathbf{w}_i^n + \Delta t \mathbf{z}_i^n\Big) \cdot \boldsymbol{\zeta}_i.
\end{aligned}
$$

The left-hand side of the above equation defines the following bilinear form associated with the structure subproblem:

$$
\begin{aligned}
a((\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}), (\boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})) :&= \rho_K h \int_\omega \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + (\Delta t)^2 a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{d}_i \cdot \boldsymbol{\xi}_i \\
&+ \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{w}_i \cdot \boldsymbol{\zeta}_i + (\Delta t)^2 a_S(\mathbf{w}, \boldsymbol{\zeta}).
\end{aligned} \tag{3.36}
$$

In order to apply the Lax-Milgram lemma we need to prove the continuity and coercivity of the bilinear form (3.36) on V_{KS}. To show that a is coercive, we write

$$
\begin{aligned}
&a((\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}), (\boldsymbol{\eta}, \mathbf{d}, \mathbf{w})) \\
&= \rho_K h \int_\omega |\boldsymbol{\eta}|^2 R + (\Delta t)^2 a_K(\boldsymbol{\eta}, \boldsymbol{\eta}) + \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} |\mathbf{d}_i|^2 \\
&+ \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i |\mathbf{w}_i|^2 + (\Delta t)^2 a_S(\mathbf{w}, \mathbf{w}) \\
&\geq c \left(\|\boldsymbol{\eta}\|^2_{L^2(R;\omega)} + \|\boldsymbol{\eta}\|^2_{H^2(\omega)} + \|\mathbf{d}\|^2_{L^2(\mathcal{N})} + \|\mathbf{w}\|^2_{L^2(\mathcal{N})} + \|\partial_s \mathbf{w}\|^2_{L^2(\mathcal{N})} \right)
\end{aligned}
$$

$$\geq c\left(\|\boldsymbol{\eta}\|^2_{H^2(\omega)} + \|\mathbf{d}\|^2_{L^2(\mathcal{N})} + \|\mathbf{w}\|^2_{H^1(\mathcal{N})}\right).$$

Now we use the condition of inextensibility and unshearability to get a bound on $\|\partial_s \mathbf{d}\|^2_{L^2(\mathcal{N})}$:

$$\|\partial_s \mathbf{d}\|_{L^2(\mathcal{N})} = \|-\mathbf{t}\times\mathbf{w}\|_{L^2(\mathcal{N})} \leq C\|\mathbf{w}\|_{L^2(\mathcal{N})}.$$

Notice how the L^2-norm of infinitesimal rotation of cross-sections keeps control over the gradient of displacement of the middle line. This now provides coercivity, i.e.

$$a((\boldsymbol{\eta},\mathbf{d},\mathbf{w}),(\boldsymbol{\eta},\mathbf{d},\mathbf{w})) \geq c\left(\|\boldsymbol{\eta}\|^2_{H^2(\omega)} + \|\mathbf{d}\|^2_{H^1(\mathcal{N})} + \|\mathbf{w}\|^2_{H^1(\mathcal{N})}\right).$$

Continuity follows immediately after employing Hölder's inequality. The Lax-Milgram lemma implies the existence of a unique solution of problem (3.34). □

Proposition 3.3 For each fixed $\Delta t > 0$, the structure subproblem (3.34) satisfies the following discrete energy equality:

$$\begin{aligned} E_N^{n+1/2} &+ \rho_K h\|\mathbf{v}^{n+1/2}-\mathbf{v}^n\|^2_{L^2(R;\omega)} + a_K(\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n,\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n) \\ &+ \rho_S\|\mathbf{k}^{n+1/2}-\mathbf{k}^n\|^2_a + \rho_S\|\mathbf{z}^{n+1/2}-\mathbf{z}^n\|^2_m \\ &+ a_S(\mathbf{w}^{n+1/2}-\mathbf{w}^n,\mathbf{w}^{n+1/2}-\mathbf{w}^n) = E_N^n, \end{aligned} \tag{3.37}$$

where

$$\|\mathbf{k}\|^2_a := \sum_{i=1}^{n_E} A_i\|\mathbf{k}_i\|_{L^2(0,l_i)}.$$

Proof. We take $(\mathbf{v}^{n+1/2},\mathbf{k}^{n+1/2},\mathbf{z}^{n+1/2})$ as a test function in the first equation in (3.34), and replace them with the corresponding expressions: $(\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n)/\Delta t$ and $(\mathbf{w}^{n+1/2}-\mathbf{w}^n)/\Delta t$ in the bilinear forms a_S and a_K, respectively, to obtain:

$$\begin{aligned} &\frac{\rho_K h}{\Delta t}\int_\omega(\mathbf{v}^{n+1/2}-\mathbf{v}^n)\cdot\mathbf{v}^{n+1/2}R + \frac{1}{\Delta t}a_K(\boldsymbol{\eta}^{n+1/2},\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n) \\ &+\frac{\rho_S}{\Delta t}\sum_{i=1}^{n_E}A_i\int_0^{l_i}(\mathbf{k}_i^{n+1/2}-\mathbf{k}_i^n)\cdot\mathbf{k}_i^{n+1/2} + \frac{\rho_S}{\Delta t}\sum_{i=1}^{n_E}\int_0^{l_i}M_i(\mathbf{z}_i^{n+1/2}-\mathbf{z}_i^n)\cdot\mathbf{z}_i^{n+1/2} \\ &+\frac{1}{\Delta t}a_S(\mathbf{w}^{n+1/2},\mathbf{w}^{n+1/2}-\mathbf{w}^n) = 0. \end{aligned}$$

We then use the algebraic identity $(a-b)\cdot a = \frac{1}{2}(|a|^2+|a-b|^2-|b|^2)$ to deal with the mixed products. After multiplying the entire equation by $2\Delta t$, the first equation in (3.34) can be written as:

$$\begin{aligned} &\rho_K h(\|\mathbf{v}^{n+1/2}\|^2+\|\mathbf{v}^{n+1/2}-\mathbf{v}^n\|^2) + a_K(\boldsymbol{\eta}^{n+1/2},\boldsymbol{\eta}^{n+1/2}) + \\ &a_K(\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n,\boldsymbol{\eta}^{n+1/2}-\boldsymbol{\eta}^n) + \rho_S(\|\mathbf{k}^{n+1/2}\|^2_a+\|\mathbf{k}^{n+1/2}-\mathbf{k}^n\|^2_a) + \\ &\rho_S(\|\mathbf{z}^{n+1/2}\|^2_m+\|\mathbf{z}^{n+1/2}-\mathbf{z}^n\|^2_m) + a_S(\mathbf{w}^{n+1/2},\mathbf{w}^{n+1/2}) + a_S(\mathbf{w}^{n+1/2}-\mathbf{w}^n, \\ &\mathbf{w}^{n+1/2}-\mathbf{w}^n) = \rho_K h\|\mathbf{v}^n\|^2 + a_K(\boldsymbol{\eta}^n,\boldsymbol{\eta}^n) + \rho_S\|\mathbf{k}^n\|^2_a + \rho_S\|\mathbf{z}^n\|^2_m + a_S(\mathbf{w}^n,\mathbf{w}^n). \end{aligned}$$

Recall that $\mathbf{u}^{n+1/2} = \mathbf{u}^n$ in this subproblem, so we can add $\rho_F \|\mathbf{u}^{n+1/2}\|^2$ on the left-hand side, and $\rho_F \|\mathbf{u}^n\|^2$ on the right-hand side of the equation, to obtain exactly the energy equality (3.37). □

3.5.2 Discrete ALE mapping

As we already mentioned, at each time step $t_n = n\Delta t,\ n = 0, \ldots, N-1$, we have a different domain Ω^n on which we have to solve the fluid subproblem. So it is not clear how to define the discrete time derivatives, since each of the functions, defined by finite differences, is defined on a different domain. For that reason we employ the Arbitrary Lagrangian-Eulerian method in order to map the current domain Ω^{n+1} to the previous one Ω^n via mapping:

$$\mathbf{A}^{n+1,n} : \Omega^{n+1} \to \Omega^n, \quad \mathbf{A}^{n+1,n} = \boldsymbol{\phi}^n(\cdot) \circ \boldsymbol{\phi}^{n+1}(\cdot)^{-1}.$$

Notice that the mapping $\mathbf{A}^{n+1,n}$ is a discrete analogue of the mapping $\mathbf{A}^{\tilde{\boldsymbol{\eta}}}$ introduced in the Section 3.4.2. We can now write the explicit formula for the mapping $\mathbf{A}^{n+1,n}$:

$$\mathbf{A}^{n+1,n} \begin{pmatrix} \tilde{z} \\ \tilde{r} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} \tilde{z} \\ \frac{R+\tilde{\eta}^n(\tilde{z},\tilde{\theta})}{R+\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})}\tilde{r} \\ \tilde{\theta} \end{pmatrix}, \tag{3.38}$$

where $(\tilde{z}, \tilde{r}, \tilde{\theta})$ denote the cylindrical coordinates in the reference domain Ω^{n+1}. Mapping $\mathbf{A}^{n+1,n}$ is a bijection, and its Jacobian is given by

$$S^{n+1,n} = |\det \nabla \mathbf{A}^{n+1,n}| = \left| \frac{R + \tilde{\eta}^n(\tilde{z}, \tilde{\theta})}{R + \tilde{\eta}^{n+1}(\tilde{z}, \tilde{\theta})} \right|^2.$$

We define the corresponding ALE domain velocity as a discrete version of the "continuous" ALE velocity $\mathbf{s}^{\tilde{\boldsymbol{\eta}}}$, i.e.

$$\mathbf{s}^{n+1,n} = \frac{\tilde{\eta}^{n+1}(\tilde{z}, \tilde{\theta}) - \tilde{\eta}^n(\tilde{z}, \tilde{\theta})}{\Delta t(R + \tilde{\eta}^{n+1}(\tilde{z}, \tilde{\theta}))} \tilde{r}\mathbf{e}_r.$$

Composite functions with this ALE mapping will be denoted by

$$\hat{\mathbf{u}}^n = \mathbf{u}^n \circ \mathbf{A}^{n+1,n}. \tag{3.39}$$

3.5.3 The semi-discretized fluid subproblem

The structure subproblem updated the position of the elastic boundary $\boldsymbol{\eta}^{n+1}$. Since we want to have the explicit formula for the ALE mapping, we are working with reparameterized displacement in the fluid subproblem. In this step the shell displacement $\tilde{\boldsymbol{\eta}}$, the mesh displacement $\mathbf{d}$ and the infinitesimal rotation $\mathbf{w}$ do not change, so we have:

$$\tilde{\boldsymbol{\eta}}^{n+1} = \tilde{\boldsymbol{\eta}}^{n+1/2}, \quad \mathbf{d}^{n+1} = \mathbf{d}^{n+1/2}, \quad \mathbf{w}^{n+1} = \mathbf{w}^{n+1/2}.$$

Furthermore, the velocity of the mesh displacement and of the infinitesimal rotation have to be zero, thus:

$$\mathbf{k}^{n+1} = \mathbf{k}^{n+1/2}, \quad \mathbf{z}^{n+1} = \mathbf{z}^{n+1/2}.$$

We define a function $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1}) \in W_F$ as a solution of the semi-discretized fluid subproblem, written in weak form:

$$\begin{aligned}
&\rho_F \int_{\Omega^{n+1}} \frac{\mathbf{u}^{n+1} - \hat{\mathbf{u}}^n}{\Delta t} \cdot \boldsymbol{\upsilon} + \frac{\rho_F}{2} \int_{\Omega^{n+1}} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}^n \cdot \boldsymbol{\upsilon}) \\
&+ \frac{\rho_F}{2} \int_{\Omega^{n+1}} \Big[((\hat{\mathbf{u}}^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\mathbf{u}^{n+1} \cdot \boldsymbol{\upsilon} - ((\hat{\mathbf{u}}^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\boldsymbol{\upsilon} \cdot \mathbf{u}^{n+1}\Big] \\
&+ 2\mu_F \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\boldsymbol{\upsilon}) + \rho_K h \int_\omega \frac{\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}}{\Delta t} \cdot \boldsymbol{\psi} R \\
&= P_{in}^n \int_{\Gamma_{in}} \upsilon_z - P_{out}^n \int_{\Gamma_{out}} \upsilon_z,
\end{aligned} \tag{3.40}$$

for all test functions $(\boldsymbol{\upsilon}, \boldsymbol{\psi}) \in V_F^{n+1} \times L^2(R;\omega)$ such that $(\boldsymbol{\upsilon} \circ \boldsymbol{\phi}^{n+1})|_\Gamma \circ \boldsymbol{\varphi} = \boldsymbol{\psi}$ on ω, where the weak solution space is defined by:

$$W_F = \{(\mathbf{u}, \mathbf{v}) \in V_F^{n+1} \times L^2(R;\omega) : (\mathbf{u} \circ \boldsymbol{\phi}^{n+1})|_\Gamma \circ \boldsymbol{\varphi} = \mathbf{v} \text{ on } \omega\},$$

with

$$V_F^{n+1} = \{\mathbf{u} \in H^1(\Omega^{n+1}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \times \mathbf{e}_z = 0 \text{ on } \Gamma_{in/out}\}.$$

The pressure terms are given by $P_{in/out}^n = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)\, dt$.

Proposition 3.4 For each fixed $\Delta t > 0$, the fluid subproblem (3.40) has a unique solution $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1}) \in W_F$.

Proof. We rewrite the first equation in (3.40) as follows:

$$\begin{aligned}
&\frac{\rho_F}{\Delta t} \int_{\Omega^{n+1}} \mathbf{u}^{n+1} \cdot \boldsymbol{\upsilon} + \frac{\rho_F}{2} \int_{\Omega^{n+1}} \Big[((\hat{\mathbf{u}}^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\mathbf{u}^{n+1} \cdot \boldsymbol{\upsilon} \\
&- ((\hat{\mathbf{u}}^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\boldsymbol{\upsilon} \cdot \mathbf{u}^{n+1}\Big] + 2\mu_F \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\boldsymbol{\upsilon}) + \frac{\rho_K h}{\Delta t} \int_\omega \mathbf{v}^{n+1} \cdot \boldsymbol{\psi} R \\
&= \frac{\rho_F}{\Delta t} \int_{\Omega^{n+1}} \hat{\mathbf{u}}^n \cdot \boldsymbol{\upsilon} + \frac{\rho_F}{2} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}^n \cdot \boldsymbol{\upsilon}) + \frac{\rho_K h}{\Delta t} \int_\omega \mathbf{v}^{n+1/2} \cdot \boldsymbol{\psi} R \\
&+ P_{in}^n \int_{\Gamma_{in}} \upsilon_z - P_{out}^n \int_{\Gamma_{out}} \upsilon_z,
\end{aligned}$$

and define the bilinear form associated with problem (3.40):

$$a((\mathbf{u}, \mathbf{v}), (\boldsymbol{\upsilon}, \boldsymbol{\psi})) = \rho_F \int_{\Omega^{n+1}} \mathbf{u} \cdot \boldsymbol{\upsilon} + \frac{\rho_F \Delta t}{2} \int_{\Omega^{n+1}} \Big[((\hat{\mathbf{u}}^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\upsilon}$$

$$
\begin{aligned}
&- ((\hat{\mathbf{u}}^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\boldsymbol{v} \cdot \mathbf{u}\Big] + 2\Delta t \mu_F \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{v}) \\
&+ \rho_K h \int_\omega \mathbf{v} \cdot \boldsymbol{\psi} R.
\end{aligned}
$$

We need to prove that this bilinear form a is coercive and continuous on W_F. To see that a is coercive, we write

$$
\begin{aligned}
a((\mathbf{u}, \mathbf{v}), (\mathbf{u}, \mathbf{v})) &= \rho_F \int_{\Omega^{n+1}} |\mathbf{u}|^2 + 2\Delta t \mu_F \int_{\Omega^{n+1}} |\mathbf{D}(\mathbf{u})|^2 + \rho_K h \int_\omega |\mathbf{v}|^2 R \\
&\geq c(\|\mathbf{u}\|^2_{L^2(\Omega^{n+1})} + \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega^{n+1})} + \|\mathbf{v}\|^2_{L^2(R;\omega)}) \\
&\geq c(\|\mathbf{u}\|^2_{H^1(\Omega^{n+1})} + \|\mathbf{v}\|^2_{L^2(R;\omega)}).
\end{aligned}
$$

To prove continuity, we apply the generalized Hölder's inequality to obtain:

$$
\begin{aligned}
a((\mathbf{u}, \mathbf{v}), (\boldsymbol{v}, \boldsymbol{\psi})) &\leq \rho_F \|\mathbf{u}\|_{L^2(\Omega^{n+1})} \|\boldsymbol{v}\|_{L^2(\Omega^{n+1})} + 2\Delta t \mu_F \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega^{n+1})} \|\mathbf{D}(\boldsymbol{v})\|_{L^2(\Omega^{n+1})} \\
&+ \rho_K h \|\mathbf{v}\|_{L^2(R;\omega)} \|\boldsymbol{\psi}\|_{L^2(R;\omega)} + \rho_F \Delta t \|\nabla \mathbf{u}\|_{L^2(\Omega^{n+1})} \|\mathbf{u}\|_{L^4(\Omega^{n+1})} \|\boldsymbol{v}\|_{L^4(\Omega^{n+1})}.
\end{aligned}
$$

Using the continuous embedding of H^1 into L^6 and furthermore continuous embedding of L^p to L^q, for $q < p$, we obtain:

$$
\begin{aligned}
a((\mathbf{u}, \mathbf{v}), (\boldsymbol{v}, \boldsymbol{\psi})) &\leq C \Big(\rho_F \|\mathbf{u}\|_{H^1(\Omega^{n+1})} \|\boldsymbol{v}\|_{H^1(\Omega^{n+1})} + 2\Delta t \mu_F \|\mathbf{u}\|_{H^1(\Omega^{n+1})} \|\boldsymbol{v}\|_{H^1(\Omega^{n+1})} \\
&+ \rho_K h \|\mathbf{v}\|_{L^2(R;\omega)} \|\boldsymbol{\psi}\|_{L^2(R;\omega)} + \rho_F \Delta t \|\mathbf{u}\|_{H^1(\Omega^{n+1})} \|\mathbf{u}\|_{H^1(\Omega^{n+1})} \|\boldsymbol{v}\|_{H^1(\Omega^{n+1})} \Big).
\end{aligned}
$$

This shows that a is continuous. The Lax-Milgram lemma now implies the existence of a unique solution $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ of problem (3.40). □

Proposition 3.5 For each fixed $\Delta t > 0$, the solution of problem (3.40) satisfies the following discrete energy inequality:

$$
\begin{aligned}
&E_N^{n+1} + \rho_F \|\mathbf{u}^{n+1} - (1-\alpha)\hat{\mathbf{u}}^n\|^2_{L^2(\Omega^{n+1})} + \rho_K h \|\mathbf{v}^{n+1} - \mathbf{v}^n\|^2_{L^2(R;\omega)} \\
&+ D_N^{n+1} \leq E_N^{n+1/2} + C\Delta t ((P_{in}^n)^2 + (P_{out}^n)^2),
\end{aligned} \tag{3.41}
$$

where α is a constant that will be specified in the proof.

Proof. We begin by replacing test function $(\boldsymbol{v}, \boldsymbol{\psi})$ by $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ in the weak formulation (3.40) to obtain:

$$
\begin{aligned}
&\frac{\rho_F}{\Delta t} \int_{\Omega^{n+1}} (\mathbf{u}^{n+1} - \hat{\mathbf{u}}^n) \cdot \mathbf{u}^{n+1} + \frac{\rho_F}{2} \int_{\Omega^{n+1}} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}^n \cdot \mathbf{u}^{n+1}) \\
&+ 2\mu_F \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\mathbf{u}^{n+1}) + \frac{\rho_K h}{\Delta t} \int_\omega (\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}) \cdot \mathbf{v}^{n+1} R \\
&= P_{in}^n \int_{\Gamma_{in}} u_z^{n+1} - P_{out}^n \int_{\Gamma_{out}} u_z^{n+1}.
\end{aligned}
$$

After applying algebraic identity $(a-b) \cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$ to take care of terms involving mixed products and multiplying the resulting equation by $2\Delta t$, we have:

$$\begin{aligned}
&\rho_F \int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 + \rho_F \int_{\Omega^{n+1}} |\mathbf{u}^{n+1} - \hat{\mathbf{u}}^n|^2 - \rho_F \int_{\Omega^{n+1}} |\hat{\mathbf{u}}^n|^2 \\
&+ \rho_F \Delta t \int_{\Omega^{n+1}} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}^n \cdot \mathbf{u}^{n+1}) + 4\Delta t \mu_F \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\mathbf{u}^{n+1}) \\
&+ \rho_K h \int_\omega |\mathbf{v}^{n+1}|^2 R + \rho_K h \int_\omega |\mathbf{v}^{n+1} - \mathbf{v}^n|^2 R - \rho_K h \int_\omega |\mathbf{v}^n|^2 R \\
&= P_{in}^n \int_{\Gamma_{in}} u_z^{n+1} - P_{out}^n \int_{\Gamma_{out}} u_z^{n+1}.
\end{aligned}$$

To get the desired energy inequality we add and subtract the term $\rho_F \int_{\Omega^n} |\mathbf{u}^n|^2$ (which corresponds to the discrete kinetic energy of the fluid at time $t_n = n\Delta t$) on the left-hand side of the equality. Recall that $S^{n+1,n} = |\det \nabla A^{n+1,n}|$ so change of variables gives us:

$$\rho_F \int_{\Omega^n} |\mathbf{u}^n|^2 = \rho_F \int_{\Omega^{n+1}} S^{n+1,n} |\hat{\mathbf{u}}^n|^2.$$

Furthermore, by using the formula for the divergence in cylindrical coordinates $\nabla \cdot f = \dfrac{\partial f_z}{\partial z} + \dfrac{1}{r}\dfrac{\partial (r f_r)}{\partial_r} + \dfrac{1}{r}\dfrac{\partial f_\theta}{\partial \theta}$, the following yields:

$$\nabla \cdot \mathbf{s}^{n+1,n} = 2\frac{\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta}) - \tilde{\eta}^n(\tilde{z},\tilde{\theta})}{\Delta t (R + \tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta}))}. \tag{3.42}$$

We put $\alpha := \dfrac{\tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta}) - \tilde{\eta}^n(\tilde{z},\tilde{\theta})}{R + \tilde{\eta}^{n+1}(\tilde{z},\tilde{\theta})}$ so that $\nabla \cdot \mathbf{s}^{n+1,n} = \dfrac{2\alpha}{\Delta t}$. For the simplicity of notation we omit ρ_F and rewrite the terms that correspond to the fluid kinetic energy (and numerical dissipation) in the following way:

$$\begin{aligned}
&\int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 + \int_{\Omega^{n+1}} |\mathbf{u}^{n+1} - \hat{\mathbf{u}}^n|^2 - \int_{\Omega^{n+1}} |\hat{\mathbf{u}}^n|^2 \\
&+ \int_{\Omega^{n+1}} \Delta t (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}^n \cdot \mathbf{u}^{n+1}) \pm \int_{\Omega^n} |\mathbf{u}^n|^2 \\
&= \int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 + \int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 - \int_{\Omega^{n+1}} 2\mathbf{u}^{n+1} \cdot \hat{\mathbf{u}}^n + \int_{\Omega^{n+1}} |\hat{\mathbf{u}}^n|^2 \\
&- \int_{\Omega^{n+1}} |\hat{\mathbf{u}}^n|^2 + \int_{\Omega^{n+1}} 2\alpha \hat{\mathbf{u}}^n \cdot \mathbf{u}^{n+1} + \int_{\Omega^{n+1}} S^{n+1,n} |\hat{\mathbf{u}}^n|^2 - \int_{\Omega^n} |\mathbf{u}^n|^2 \\
&= \int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 - \int_{\Omega^n} |\mathbf{u}^n|^2 + \int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 \\
&- \int_{\Omega^{n+1}} 2(1-\alpha)\mathbf{u}^{n+1} \cdot \hat{\mathbf{u}}^n + \int_{\Omega^{n+1}} S^{n+1,n} |\hat{\mathbf{u}}^n|^2 \\
&= \int_{\Omega^{n+1}} |\mathbf{u}^{n+1}|^2 - \int_{\Omega^n} |\mathbf{u}^n|^2 + \int_{\Omega^{n+1}} |\mathbf{u}^{n+1} - (1-\alpha)\hat{\mathbf{u}}^n|^2 \\
&- \int_{\Omega^{n+1}} |(1-\alpha)\hat{\mathbf{u}}^n|^2 + \int_{\Omega^{n+1}} S^{n+1,n} |\hat{\mathbf{u}}^n|^2.
\end{aligned}$$

First two terms on the right-hand side correspond to the discrete kinetic energy at time t_{n+1} and t_n, respectively, while the third term corresponds to the numerical dissipation.

By a simple calculation we obtain that $S^{n+1,n} - (1-\alpha)^2 = 0$ so the last two terms cancel. After the estimate of the pressure terms, we obtain the following inequality:

$$\begin{aligned}
&\rho_F \left(\|\mathbf{u}^{n+1}\|^2_{L^2(\Omega^{n+1})} + \|\mathbf{u}^{n+1} - (1-\alpha)\hat{\mathbf{u}}^n\|^2_{L^2(\Omega^{n+1})} \right) + 2\Delta t \mu_F \|\mathbf{D}(\mathbf{u}^{n+1})\|^2_{L^2(\Omega^{n+1})} \\
&\quad + \rho_K h \|\mathbf{v}^{n+1}\|^2 + \rho_K h \|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2 \\
&\leq \rho_F \|\mathbf{u}^n\|_{L^2(\Omega^n)} + \rho_K h \|\mathbf{v}^{n+1/2}\|^2 + C\Delta t \left((P^n_{in})^2 + (P^n_{out})^2 \right).
\end{aligned}$$

It is interesting to notice how the presence of nonlinear advection term together with adding (and subtracting) the term $\rho_F \int_{\Omega^n} |\mathbf{u}^n|^2$ makes the discrete kinetic energy of the fluid subproblem to be decreasing in time, and to thus satisfy the desired energy estimate.

Finally, recall that $\boldsymbol{\eta}^{n+1} = \boldsymbol{\eta}^{n+1/2}$ and $\mathbf{w}^{n+1} = \mathbf{w}^{n+1/2}$ in the fluid subproblem, so we can add $a_K(\boldsymbol{\eta}^{n+1}, \boldsymbol{\eta}^{n+1})$ and $a_S(\mathbf{w}^{n+1}, \mathbf{w}^{n+1})$ on the left-hand side, and $a_K(\boldsymbol{\eta}^{n+1/2}, \boldsymbol{\eta}^{n+1/2})$ and $a_S(\mathbf{w}^{n+1/2}, \mathbf{w}^{n+1/2})$ on the right-hand side of the previous inequality. Furthermore, since $\mathbf{k}^{n+1} = \mathbf{k}^{n+1/2}$ and $\mathbf{z}^{n+1} = \mathbf{z}^{n+1/2}$, we add $\|\mathbf{k}^{n+1}\|^2_a$ and $\|\mathbf{z}^{n+1}\|^2_m$ on the left-hand side, and $\|\mathbf{k}^{n+1/2}\|^2_a$ and $\|\mathbf{z}^{n+1/2}\|^2_m$ on the right-hand side, to obtain exactly the energy inequality (3.41).

$\square$

3.6 Uniform energy estimates

Our goal is to show that there exist subsequences of functions, parameterized by N (which depends on Δt), defined by the time-discretization via Lie-splitting specified above, which converge to a weak solution of Problem 1. We show that the sequence of approximations defined above is uniformly bounded (with respect to Δt) in energy norm.

Theorem 3.6 Let $\Delta t > 0$ and $N = T/\Delta t$. Furthermore, let $E^{n+1/2}_N, E^{n+1}_N$ and D^{n+1}_N be the kinetic energy and dissipation given by (3.32) and (3.33), respectively. If there exists a time $T > 0$ such that for every $t \leq T$, $\Gamma^{\boldsymbol{\eta}}(t)$ remains a subgraph of a function, and if the injectivity of the mapping $\mathbf{A}^{\tilde{\boldsymbol{\eta}}}$ is ensured, i.e. function $\mathbf{g}$ defined in (3.28) is injective, then there exists a constant $K > 0$ independent of Δt (and N) such that the following estimates hold:

1. $E^{n+1/2}_N \leq K, E^{n+1}_N \leq K, \forall n = 0, 1, \ldots, N-1$,

2. $\sum_{n=0}^{N-1} D^{n+1}_N \leq K$,

3. $\sum_{n=0}^{N-1} \left(\rho_F \|\mathbf{u}^{n+1} - (1-\alpha)\hat{\mathbf{u}}^n\|^2 + \rho_K h (\|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2 + \|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2) \right) \leq K$,

4. $\sum_{n=0}^{N-1} \rho_S \left(\|\mathbf{k}^{n+1} - \mathbf{k}^n\|^2_a + \|\mathbf{z}^{n+1} - \mathbf{z}^n\|^2_m \right) \leq K$,

5. $\sum_{n=0}^{N-1} a_K(\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n) \leq K$,

$\sum_{n=0}^{N-1} a_S(\mathbf{w}^{n+1} - \mathbf{w}^n, \mathbf{w}^{n+1} - \mathbf{w}^n) \leq K$.

Proof. We begin by adding the energy estimates (3.37) and (3.41) to obtain:

$$\begin{aligned}
&E_N^{n+1/2} + \rho_K h\|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2 + a_K(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n)\\
&+ \rho_S\|\mathbf{k}^{n+1/2} - \mathbf{k}^n\|_a^2 + \rho_S\|\mathbf{z}^{n+1/2} - \mathbf{z}^n\|_m^2\\
&+ a_S(\mathbf{w}^{n+1/2} - \mathbf{w}^n, \mathbf{w}^{n+1/2} - \mathbf{w}^n) + E_N^{n+1} + \rho_F\|\mathbf{u}^{n+1} - (1-\alpha)\hat{\mathbf{u}}^n\|^2\\
&+ D_N^{n+1} + \rho_K h\|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2 \leq E_N^n + E_N^{n+1/2} + C\Delta t((P_{in}^n)^2 + (P_{out}^n)^2).
\end{aligned}$$

We take the sum from $n = 0$ to $n = N - 1$ on both sides to obtain:

$$\begin{aligned}
&E_N^N + \sum_{n=0}^{N-1} D_N^{n+1} + \sum_{n=0}^{N-1}\Big(\rho_F\|\mathbf{u}^{n+1} - (1-\alpha)\hat{\mathbf{u}}^n\|^2 + \rho_K h\|\mathbf{v}^{n+1} - \mathbf{v}^{n+1/2}\|^2\\
&+ \rho_K h\|\mathbf{v}^{n+1/2} - \mathbf{v}^n\|^2 + \rho_S\|\mathbf{k}^{n+1/2} - \mathbf{k}^n\|_a^2 + \rho_S\|\mathbf{z}^{n+1/2} - \mathbf{z}^n\|_m^2\\
&+ a_K(\boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n, \boldsymbol{\eta}^{n+1/2} - \boldsymbol{\eta}^n) + a_S(\mathbf{w}^{n+1/2} - \mathbf{w}^n, \mathbf{w}^{n+1/2} - \mathbf{w}^n)\Big)\\
&\leq E_0 + C\Delta t\sum_{n=0}^{N-1}\left((P_{in}^n)^2 + (P_{out}^n)^2\right).
\end{aligned}$$

The term involving the inlet and outlet pressure data can be easily estimated by recalling that on each subinterval (t_n, t_{n+1}) the pressure data is approximated by a constant, which is equal to the average value of the pressure over that time interval. Therefore, after using Hölder's inequality, we have:

$$\begin{aligned}
\Delta t\sum_{n=0}^{N-1}(P_{in}^n)^2 &= \Delta t\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in}(t)\,dt\right)^2\\
&= \frac{1}{\Delta t}\sum_{n=0}^{N-1}\left(\int_{n\Delta t}^{(n+1)\Delta t} P_{in}(t)\,dt\right)^2\\
&\leq \frac{1}{\Delta t}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P_{in}^2(t)\,dt\int_{n\Delta t}^{(n+1)\Delta t} 1\,dt\\
&= \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P_{in}^2(t)\,dt = \|P_{in}\|_{L^2(0,T)}^2.
\end{aligned}$$

By using the pressure estimate to bound the right-hand side in the above energy estimate, we have obtained all the statements of the Theorem, with the constant K given by $K = E_0 + C\|P_{in/out}\|_{L^2(0,T)}^2$. $\square$

3.7 Weak convergence of the approximate solutions

The approach described above defines a set of discrete values in time, which can be used to define the set of approximate solutions on $(0,T)$. Indeed, we define approximate solutions to be the piecewise constant functions on each subinterval $((n-1)\Delta t, n\Delta t], n = 1,2,\ldots,N$, such that for $t \in ((n-1)\Delta t, n\Delta t], n = 1,2,\ldots,N$ we have:

$$\begin{aligned}
&\mathbf{u}_N(t,\cdot) = \mathbf{u}_N^n,\ \boldsymbol{\eta}_N(t,\cdot) = \boldsymbol{\eta}_N^n,\ \tilde{\boldsymbol{\eta}}_N(t,\cdot) = \tilde{\boldsymbol{\eta}}_N^n,\\
&\mathbf{v}_N(t,\cdot) = \mathbf{v}_N^n,\ \mathbf{v}_N^*(t,\cdot) = \mathbf{v}_N^{n-1/2},\ \mathbf{d}_N(t,\cdot) = \mathbf{d}_N^n,\\
&\mathbf{w}_N(t,\cdot) = \mathbf{w}_N^n,\ \mathbf{k}_N(t,\cdot) = \mathbf{k}_N^n,\ \mathbf{z}_N(t,\cdot) = \mathbf{z}_N^n.
\end{aligned} \tag{3.43}$$

Notice that we used $\mathbf{v}_N^*(t,\cdot) = \mathbf{v}_N^{n-1/2}$ to denote the approximate shell velocity functions determined by the structure subproblem, and $\mathbf{v}_N(t,\cdot) = \mathbf{v}_N^n$ to denote the approximate shell velocity functions determined by the fluid subproblem. Consequently, these are not necessarily the same. However, later we will show that $\|\mathbf{v}_N - \mathbf{v}_N^*\|_{L^2(R;\omega)} \to 0$ as $\Delta t \to 0$. Using Theorem 3.6 we now show that these sequences are uniformly bounded in the appropriate solution spaces.

Due to the fact that at each time step we deal with a problem defined on a different domain, the uniform energy estimates for the fluid velocity in Theorem 3.6 give us boundedness of $\mathbf{u}_N^n$ in $L^\infty(0,T;L^2(\Omega^n))$, for each $n = 1,2,\ldots,N$. Since we need to make sense of the limit of the approximate velocity $\mathbf{u}_N = (\mathbf{u}_N^1,\ldots,\mathbf{u}_N^N)$ when $N \to \infty$, we introduce the maximal fluid domain Ω_M which is a cylinder of radius R_{max} and define (constant) extensions of the approximate solutions in the following way:

$$\tilde{\mathbf{u}}_N^n(z,r,\theta) = \begin{cases} \mathbf{u}_N^n(z,r,\theta), & (z,r,\theta) \text{ in } \Omega^n,\\ \mathbf{u}_N^n(z,R+\tilde{\eta}^n,\theta), & (z,r,\theta) \text{ in } \Omega_M\backslash\Omega^n. \end{cases} \tag{3.44}$$

The maximal fluid domain is such that contains all the time-dependent domains, i.e. $\Omega(t) \subset \Omega_M, \forall t \in [0,T]$. Throughout the rest of this chapter, we intend to work only with reparameterized domains, so we omitted $\sim$ from the cylindrical coordinates in order to keep the notation more clear.

We are now in position to prove the following proposition:

Proposition 3.7 The sequence $(\tilde{\mathbf{u}}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega_M))$.

Proof. From the definition of the extended sequence $\tilde{\mathbf{u}}_N$ we have:

$$\begin{aligned}
\|\tilde{\mathbf{u}}_N(t)\|^2_{L^2(\Omega_M)} &= \sum_{n=1}^N \|\tilde{\mathbf{u}}_N^n\|^2_{L^2(\Omega_M)}\\
&= \sum_{n=1}^N \left(\|\mathbf{u}_N^n\|^2_{L^2(\Omega^n)} + \|\mathbf{u}_N^n(z,R+\tilde{\eta}_N^n,\theta)\|^2_{L^2(\Omega_M\backslash\Omega^n)}\right)
\end{aligned}$$

$$= \sum_{n=1}^{N} \left(\|\mathbf{u}_N^n\|^2_{L^2(\Omega^n)} + C\|\mathbf{v}_N^n\|^2_{L^2(R;\omega)} \right),$$

where $C = R_{max} - (R + \tilde{\eta}_N^n)$ is a positive constant. Using Theorem 3.6, precisely Statement 1, we obtain the desired result. □

Proposition 3.8 The sequence $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;V_K)$ and the sequence $(\mathbf{w}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H^1(\mathcal{N}))$.

Proof. The first statement of Theorem 3.6 states that $E_N^{n+1} \le K,\ \forall n = 0,\dots,N-1$, which implies

$$\|\boldsymbol{\eta}_N(t)\|^2_{H^2(\omega)} \le a_K(\boldsymbol{\eta}_N(t), \boldsymbol{\eta}_N(t)) \le K, \quad \forall t \in [0,T].$$

Therefore,

$$\|\boldsymbol{\eta}_N\|_{L^\infty(0,T;V_K)} \le K.$$

The boundedness of the sequence $(\mathbf{w}_N)_{N\in\mathbb{N}}$ also follows from the first statement of Theorem 3.6. Namely, we have

$$\|\mathbf{w}_N(t)\|^2_{L^2(\mathcal{N})} \le \|\mathbf{w}_N(t)\|^2_m \le K,$$
$$\|\partial_s \mathbf{w}_N(t)\|^2_{L^2(\mathcal{N})} \le a_S(\mathbf{w}_N(t), \mathbf{w}_N(t)) \le K,$$

which gives us the desired bound. □

Notice that from the uniform energy estimates we do not get any bounds on the sequence $(\mathbf{d}_N)_{N\in\mathbb{N}}$. However, using the condition of inextensibility and unshearability, together with the regularity of $\mathbf{w}_N$, we can easily prove the following result on the boundedness of the sequence $(\mathbf{d}_N)_{N\in\mathbb{N}}$.

Corollary 3.9 The sequence $(\mathbf{d}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H^1(\mathcal{N}))$.

The following uniform bounds for the shell and mesh approximate velocities are a direct consequence of Theorem 3.6.

Proposition 3.10 The following statements hold:

(i) $(\mathbf{v}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(R;\omega))$,
$(\mathbf{v}^*_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(R;\omega))$,

(ii) $(\mathbf{k}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathcal{N}))$,
$(\mathbf{z}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathcal{N}))$.

To pass to the limit in the weak formulation of approximate solutions, we need additional regularity in time of the sequences $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}, (\mathbf{d}_N)_{N\in\mathbb{N}}$ and $(\mathbf{w}_N)_{N\in\mathbb{N}}$. For this purpose, we introduce a slightly different set of approximate functions. Namely, for each fixed Δt, define $\overline{\boldsymbol{\eta}}_N, \overline{\mathbf{d}}_N$ and $\overline{\mathbf{w}}_N$ to be continuous, *linear* on each subinterval $[(n-1)\Delta t, n\Delta t],\ n = 1, \ldots, N$, and such that

$$\begin{aligned}
&\overline{\boldsymbol{\eta}}_N(n\Delta t, \cdot) = \boldsymbol{\eta}_N(n\Delta t, \cdot), \overline{\mathbf{v}}_N(n\Delta t, \cdot) = \mathbf{v}_N(n\Delta t, \cdot), \\
&\overline{\mathbf{d}}_N(n\Delta t, \cdot) = \mathbf{d}_N(n\Delta t, \cdot), \overline{\mathbf{w}}_N(n\Delta t, \cdot) = \mathbf{w}_N(n\Delta t, \cdot), \\
&\overline{\mathbf{k}}_N(n\Delta t, \cdot) = \mathbf{k}_N(n\Delta t, \cdot), \overline{\mathbf{z}}_N(n\Delta t, \cdot) = \mathbf{z}_N(n\Delta t, \cdot).
\end{aligned} \tag{3.45}$$

We now observe:

$$\partial_t \overline{\boldsymbol{\eta}}_N(t) = \frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t} = \frac{\boldsymbol{\eta}_N^{n+1/2} - \boldsymbol{\eta}_N^n}{\Delta t} = \mathbf{v}_N^{n+1/2}, \quad t \in (n\Delta t, (n+1)\Delta t].$$

Since $\mathbf{v}_N^*$ is a piecewise constant function, as defined before via $\mathbf{v}_N^*(t, \cdot) = \mathbf{v}_N^{n+1/2}$, for $t \in (n\Delta t, (n+1)\Delta t]$, we see that

$$\partial_t \overline{\boldsymbol{\eta}}_N = \mathbf{v}_N^* \text{ a.e. on } (0, T). \tag{3.46}$$

From (3.46), and from the uniform boundedness of $E_N^{n+i/2}$ provided by Theorem 3.6, we obtain the uniform boundedness of $(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ in $W^{1,\infty}(0, T; L^2(R; \omega))$. Now, since sequences $(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ and $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ have the same limit (distributional limit is unique), we get that the weak* limit of $\boldsymbol{\eta}_N$ is, in fact, in $W^{1,\infty}(0, T; L^2(R; \omega))$.

Using analogous arguments, one also obtains that the weak* limits of $(\mathbf{d}_N)_{N\in\mathbb{N}}$ and $(\mathbf{w}_N)_{N\in\mathbb{N}}$ are in $W^{1,\infty}(0, T; L^2(\mathcal{N}))$. This is because the corresponding velocity approximations are uniformly bounded in the corresponding norms, as stated in Statement 4 of Theorem 3.6.

From the uniform boundedness of approximate sequences we can now conclude that for each approximate sequence there exists a subsequence which, with a slight abuse of notation, we denote the same way as the original sequence, and which converges weakly and weakly*, depending on the function space. More precisely, the following result holds:

Lemma 3.11 There exist subsequences $(\tilde{\mathbf{u}}_N)_{N\in\mathbb{N}}, (\boldsymbol{\eta}_N)_{N\in\mathbb{N}}, (\mathbf{v}_N)_{N\in\mathbb{N}}, (\mathbf{v}_N^*)_{N\in\mathbb{N}}$, $(\mathbf{d}_N)_{N\in\mathbb{N}}, (\mathbf{w}_N)_{N\in\mathbb{N}}, (\mathbf{k}_N)_{N\in\mathbb{N}}, (\mathbf{z}_N)_{N\in\mathbb{N}}$, and the functions $\tilde{\mathbf{u}} \in L^\infty(0, T; L^2(\Omega_M))$, $\boldsymbol{\eta} \in L^\infty(0, T; V_K) \cap W^{1,\infty}(0, T; L^2(R; \omega)), \mathbf{d}, \mathbf{w} \in L^\infty(0, T; H^1(\mathcal{N})) \cap W^{1,\infty}(0, T; L^2(\mathcal{N}))$,

$\mathbf{v}, \mathbf{v}^* \in L^\infty(0,T;L^2(R;\omega))$, and $\mathbf{k}, \mathbf{z} \in L^\infty(0,T;L^2(\mathcal{N}))$, such that

$$
\begin{aligned}
&\tilde{\mathbf{u}}_N \rightharpoonup \tilde{\mathbf{u}} \text{ weakly* in } L^\infty(0,T;L^2(\Omega_M)),\\
&\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta} \text{ weakly* in } L^\infty(0,T;V_K),\\
&\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta} \text{ weakly* in } W^{1,\infty}(0,T;L^2(R;\omega)),\\
&\mathbf{d}_N \rightharpoonup \mathbf{d} \text{ weakly* in } L^\infty(0,T;H^1(\mathcal{N})),\\
&\mathbf{d}_N \rightharpoonup \mathbf{d} \text{ weakly* in } W^{1,\infty}(0,T;L^2(\mathcal{N})),\\
&\mathbf{w}_N \rightharpoonup \mathbf{w} \text{ weakly* in } L^\infty(0,T;H^1(\mathcal{N})),\\
&\mathbf{w}_N \rightharpoonup \mathbf{w} \text{ weakly* in } W^{1,\infty}(0,T;L^2(\mathcal{N})),\\
&\mathbf{v}_N \rightharpoonup \mathbf{v} \text{ weakly* in } L^\infty(0,T;L^2(R;\omega)),\\
&\mathbf{v}^*_N \rightharpoonup \mathbf{v}^* \text{ weakly* in } L^\infty(0,T;L^2(R;\omega)),\\
&\mathbf{k}_N \rightharpoonup \mathbf{k} \text{ weakly* in } L^\infty(0,T;L^2(\mathcal{N})),\\
&\mathbf{z}_N \rightharpoonup \mathbf{z} \text{ weakly* in } L^\infty(0,T;L^2(\mathcal{N})).
\end{aligned}
$$

Furthermore,

$$\mathbf{v} = \mathbf{v}^*.$$

Proof. We only need to show that $\mathbf{v} = \mathbf{v}^*$. To show this, we use the definition of approximate sequences as step functions in t, i.e.

$$
\begin{aligned}
\|\mathbf{v}_N - \mathbf{v}^*_N\|^2_{L^2(0,T;L^2(R;\omega))} &= \int_0^T \|\mathbf{v}_N - \mathbf{v}^*_N\|^2_{L^2(R;\omega)}\, dt\\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\mathbf{v}_N^{n+1} - \mathbf{v}_N^{n+1/2}\|^2_{L^2(R;\omega)}\, dt\\
&= \sum_{n=0}^{N-1} \|\mathbf{v}_N^{n+1} - \mathbf{v}_N^{n+1/2}\|^2_{L^2(R;\omega)}\, \Delta t \le K\Delta t.
\end{aligned}
$$

The last inequality follows from Statement 3 of Theorem 3.6. By letting $\Delta t \to 0$, we get that $\mathbf{v} = \mathbf{v}^*$. □

3.8 Strong convergence of approximate sequences

To show that the limits obtained in the previous section satisfy the weak form of Problem 1, we need to show that our sequences converge strongly in the appropriate function spaces.

3.8.1 Strong convergence of the shell displacement

Recall that from Lemma 3.11 we have that

$$(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;V_K)\cap W^{1,\infty}(0,T;L^2(R;\omega)).$$

We use interpolation of Sobolev spaces (Theorem 4.17 from [1]) to obtain the following inequality:

$$\begin{aligned}\|\overline{\boldsymbol{\eta}}_N(t+\Delta t)-\overline{\boldsymbol{\eta}}_N(t)\|_{H^{2\alpha}(\omega)} &\le C\|\overline{\boldsymbol{\eta}}_N(t+\Delta t)-\overline{\boldsymbol{\eta}}_N(t)\|^{1-\alpha}_{L^2(\omega)}\|\overline{\boldsymbol{\eta}}_N(t+\Delta t)-\overline{\boldsymbol{\eta}}_N(t)\|^{\alpha}_{H^2(\omega)}\\ &\le C(\Delta t)^{1-\alpha},\ \text{where } 0<\alpha<1,\end{aligned} \tag{3.47}$$

i.e. we get that $(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $C^{0,1-\alpha}([0,T];H^{2\alpha}),\ 0<\alpha<1$. Now, from the continuous embedding of $H^{2\alpha}$ into $H^{2\alpha-\varepsilon}$, and by applying the Arzelà-Ascoli theorem, we conclude that $(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ has a convergent subsequence, which we again denote by $(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ such that

$$\overline{\boldsymbol{\eta}}_N \to \boldsymbol{\eta} \text{ in } C([0,T];H^s(\omega)),\ 0<s<2.$$

Here, we used the fact that the sequences $(\overline{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ i $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ have the same limit. We now prove the following result:

Proposition 3.12 $\boldsymbol{\eta}_N \to \boldsymbol{\eta}$ in $L^\infty(0,T;C(\overline{\omega}))$.

Proof. First, we prove that $\boldsymbol{\eta}_N \to \boldsymbol{\eta}$ in $L^\infty(0,T;H^s(\omega))$, for $0<s<2$. That result follows from the continuity in time of $\boldsymbol{\eta}$, and from the fact that $\overline{\boldsymbol{\eta}} \to \boldsymbol{\eta}$ in $C([0,T];H^s(\omega))$, for $0<s<2$. Namely, we write

$$\begin{aligned}\|\boldsymbol{\eta}_N(t)-\boldsymbol{\eta}(t)\|_{H^s(\omega)} &= \|\boldsymbol{\eta}_N(t)-\boldsymbol{\eta}(n\Delta t)+\boldsymbol{\eta}(n\Delta t)-\boldsymbol{\eta}(t)\|_{H^s(\omega)}\\ &= \|\boldsymbol{\eta}_N(n\Delta t)-\boldsymbol{\eta}(n\Delta t)+\boldsymbol{\eta}(n\Delta t)-\boldsymbol{\eta}(t)\|_{H^s(\omega)}\\ &\le \|\boldsymbol{\eta}_N(n\Delta t)-\boldsymbol{\eta}(n\Delta t)\|_{H^s(\omega)}+\|\boldsymbol{\eta}(n\Delta t)-\boldsymbol{\eta}(t)\|_{H^s(\omega)}\\ &= \|\overline{\boldsymbol{\eta}}_N(n\Delta t)-\boldsymbol{\eta}(n\Delta t)\|_{H^s(\omega)}+\|\boldsymbol{\eta}(n\Delta t)-\boldsymbol{\eta}(t)\|_{H^s(\omega)}\\ &< \varepsilon.\end{aligned}$$

Here, we used the fact that for $t\in((n-1)\Delta t,n\Delta t]$ it follows $\overline{\boldsymbol{\eta}}_N(n\Delta t)=\boldsymbol{\eta}_N(n\Delta t)=\boldsymbol{\eta}_N(t)$. Furthermore, for $s>1$ we have $H^s(\omega)\hookrightarrow C(\overline{\omega})$, so we can conclude the proof. □

However, the spatial regularity of the sequence $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ that we obtained in the previous proposition, will not be sufficient to pass to the limit, or even to apply some well known results. So we have to assume some stronger convergence properties of approximate solution for the shell displacement, i.e. we will be assuming that

$$\|\boldsymbol{\eta}_N\|_{C([0,T];W^{1,\infty}(\omega))} \leq C,$$

i.e. that $(\boldsymbol{\eta}_N)_{N\in\mathbb{N}}$ is a sequence of uniformly Lipschitz functions. In particular, we have that $\|\boldsymbol{\eta}_N\|_{W^{1,\infty}(\omega)} \leq C$. By applying the same procedure as in the proof of Theorem 5.5-1 from [29], we can prove that the mappings $\mathbf{id} + \boldsymbol{\eta}_N$ are injective. This implies the injectivity of function $\mathbf{g}$ defined in (3.28) and ensures that the reparameterizations $\tilde{\boldsymbol{\eta}}_N$ are well defined. Now we have that the sequence $\mathbf{id} + \boldsymbol{\eta}_N$ is a sequence of injective, uniformly Lipschitz functions. Furthermore, from $\|\boldsymbol{\eta}_N\|_{W^{1,\infty}(\omega)} \leq C$, we have that the gradient of $\mathbf{id} + \boldsymbol{\eta}_N$ is bounded from below, which implies that the gradient of $(\mathbf{id} + \boldsymbol{\eta}_N)^{-1}$ is bounded from above, i.e. $(\mathbf{id} + \boldsymbol{\eta}_N)^{-1}$ is a sequence of uniformly Lipschitz functions. Finally, we have that the sequence $\mathbf{id} + \boldsymbol{\eta}_N$ is a sequence of uniformly bi-Lipschitz functions.

Using the previous conclusions and the definition of the reparameterized shell displacement, we can easily prove the following proposition:

Proposition 3.13 The sequence $(\tilde{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}$ is a sequence of uniformly bi-Lipschitz functions, i.e. sequences $(\tilde{\boldsymbol{\eta}}_N)$ and $(\tilde{\boldsymbol{\eta}}_N^{-1})$ are bounded in $C([0,T];W^{1,\infty}(\omega))$.

3.8.2 Convergence of the gradients

We need to additionally show that the sequence of the gradients of the fluid velocity converges weakly to the gradient of the limiting velocity in order to be able to pass to the limit in the weak formulation of the fluid-mesh-shell interaction problem. From Theorem 3.6, we know that the symmetrized gradient is bounded in the following way:

$$\sum_{n=0}^{N-1} \int_{\Omega^{n+1}} |\mathbf{D}(\mathbf{u}_N^{n+1})|^2 \Delta t \leq K.$$

Since we are not able (at least for now) to show the boundedness of the gradient of the extended sequence of the approximate fluid velocity $\nabla \tilde{\mathbf{u}}_N$, we map the approximate fluid velocities and the limiting fluid velocity onto the corresponding physical domains. For this purpose, for $n = 0, \ldots, N-1$, we introduce the following functions which are defined on the maximal domain Ω_M :

$$\chi_N(t,\mathbf{x}) = \begin{cases} 1, & t \in (n\Delta t, (n+1)\Delta t],\ \mathbf{x} \in \Omega^{n+1}, \\ 0, & \text{otherwise}\ , \end{cases} \tag{3.48}$$

$$\chi(t,\mathbf{x}) = \begin{cases} 1, & t \in (0,T],\ \mathbf{x} \in \Omega^{\tilde{\boldsymbol{\eta}}}(t), \\ 0, & \text{otherwise}\ , \end{cases}$$

where $\tilde{\boldsymbol{\eta}}$ is the weak* limit of $\tilde{\boldsymbol{\eta}}_N$ in $L^\infty(0,T;W^{1,\infty}(\omega))$. First we show the boundedness of the $\chi_N \nabla \tilde{\mathbf{u}}_N$ in order to get weakly convergent subsequence:

$$\begin{aligned}\int_0^T \|\chi_N \nabla \tilde{\mathbf{u}}_N\|^2_{L^2(\Omega_M)} &= \sum_{n=0}^{N-1} \|\nabla \mathbf{u}_N^{n+1}\|^2_{L^2(\Omega^{n+1})}\Delta t \\ &\leq C(\Omega^{n+1}) \sum_{n=0}^{N-1} \|\mathbf{D}(\mathbf{u}_N^{n+1})\|^2_{L^2(\Omega^{n+1})}\Delta t,\end{aligned}$$

where $C(\Omega^{n+1})$ is a constant from Korn's inequality that depends on $\Omega^{n+1},\ n = 0,\dots,N-1$. Since the approximate reparameterized shell displacements are uniformly bi-Lipschitz functions, from Lemma 1 in [65] we obtain the existence of a universal Korn constant for all the domains, i.e. there exists a constant D such that

$$C(\Omega^{n+1}) \leq D, \quad ,n = 0,\dots,N-1.$$

Finally,

$$\int_0^T \|\chi_N \nabla \tilde{\mathbf{u}}_N\|^2_{L^2(\Omega_M)} \leq D \sum_{n=0}^{N-1} \|\mathbf{D}(\mathbf{u}_N^{n+1})\|^2_{L^2(\Omega^{n+1})}\Delta t \leq DK,$$

which implies that the sequence $(\chi_N \nabla \tilde{\mathbf{u}}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega_M))$. We can furthermore conclude that there exists a subsequence, which we denote the same way, and a function $\mathbf{G} \in L^2(0,T;L^2(\Omega_M))$ such that

$$\chi_N \nabla \tilde{\mathbf{u}}_N \rightharpoonup \mathbf{G} \text{ weakly in } L^2(0,T;L^2(\Omega_M)),$$

i.e.

$$\lim_{N\to\infty} \int_0^T \int_{\Omega_M} \chi_N \nabla \tilde{\mathbf{u}}_N \cdot \boldsymbol{v} = \int_0^T \int_{\Omega_M} \mathbf{G} \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in C_c^\infty((0,T)\times\Omega_M).$$

We want to show that $\mathbf{G}$, which is a weak limit, is equal to the $\chi \nabla \tilde{\mathbf{u}}$. In order to do that, we first consider the set $\Omega_M \backslash \Omega^{\tilde{\boldsymbol{\eta}}}(t)$ and show that $\mathbf{G} = 0$ there, and then the set $\Omega^{\tilde{\boldsymbol{\eta}}}(t)$ and show that $\mathbf{G} = \nabla \mathbf{u}$ there.

Take a test function $\boldsymbol{v}_1$ such that $\operatorname{supp} \boldsymbol{v}_1 \subset (0,T)\times(\Omega_M \backslash \Omega^{\tilde{\boldsymbol{\eta}}}(t))$. Using the uniform convergence of the sequence $\tilde{\boldsymbol{\eta}}_N$, we can find N_1 such that $\chi_N(\mathbf{x}) = \chi(\mathbf{x}) = 0, N \geq N_1, \mathbf{x} \in \operatorname{supp} \boldsymbol{v}_1$. Therefore, we have

$$\int_0^T \int_{\Omega_M} \mathbf{G} \cdot \boldsymbol{v}_1 = \lim_{N\to\infty} \int_0^T \int_{\Omega_M} \chi_N \nabla \tilde{\mathbf{u}}_N \cdot \boldsymbol{v}_1 = 0.$$

Thus, $\mathbf{G} = 0$ on $(0,T)\times(\Omega_M \backslash \Omega^{\tilde{\boldsymbol{\eta}}}(t))$.

For the second part of the proof, we take a test function $\boldsymbol{v}_2$ such that $\operatorname{supp} \boldsymbol{v}_2 \subset (0,T)\times\Omega^{\tilde{\boldsymbol{\eta}}}(t)$. Using the uniform convergence of the sequence $\tilde{\boldsymbol{\eta}}_N$, we can find N_2 such that $\chi_N(\mathbf{x}) = \chi(\mathbf{x}) = 1, N \geq N_2, \mathbf{x} \in \operatorname{supp} \boldsymbol{v}_2$. Therefore, we have

$$\int_0^T \int_{\Omega_M} \mathbf{G} \cdot \boldsymbol{v}_2 = \lim_{N\to\infty} \int_0^T \int_{\Omega_M} \chi_N \nabla \tilde{\mathbf{u}}_N \cdot \boldsymbol{v}_2 = \int_0^T \int_{\Omega^{\tilde{\eta}}(t)} \nabla \mathbf{u} \cdot \boldsymbol{v}_2.$$

Thus, $\mathbf{G} = \nabla \mathbf{u}$ on $(0,T) \times \Omega^{\tilde{\eta}}(t)$. Therefore, we have shown that

$$\chi_N \nabla \tilde{\mathbf{u}}_N \rightharpoonup \chi \nabla \tilde{\mathbf{u}} \quad \text{in} \quad L^2(0,T; L^2(\Omega_M)). \tag{3.49}$$

3.8.3 Strong convergence of velocities

In this section, we will apply the main result from [62] to obtain the strong convergence of the approximate fluid, mesh and shell velocities. For the completeness of the proof, we first state this result.

Theorem 3.14 ([62]) Let V and H be Hilbert spaces such that $V \subset\subset H$. Suppose that $\{\mathbf{u}_N\} \subset L^2(0,T;H)$ is a sequence such that $\mathbf{u}_N(t,\cdot) = \mathbf{u}_N^n(\cdot)$ on $(n-1)\Delta t, n\Delta t$, $n = 1,\dots,N$ with $N\Delta t = T$. Let V_N^n and Q_N^n be Hilbert spaces such that $(V_N^n, Q_N^n) \hookrightarrow V \times V$, where the embeddings are uniformly continuous with respect to N and n, and $V_N^n \subset\subset \overline{Q_N^n}^H \hookrightarrow (Q_N^n)'$. Let $\mathbf{u}_N^n \in V_N^n$, $n = 1,\dots,N$. If the following is true:

(A) There exists a universal constant $C > 0$ such that for every N

 (A1) $\sum_{n=1}^N \|\mathbf{u}_N^n\|^2_{V_N^n} \Delta t \leq C$,

 (A2) $\|\mathbf{u}_N\|_{L^\infty(0,T;H)} \leq C$,

 (A3) $\|\tau_{\Delta t}\mathbf{u}_N - \mathbf{u}_N\|_{L^2(\Delta t,T;H)} \leq C\Delta t$,

 where $\tau_{\Delta t}\mathbf{u}_N(t,\cdot) = \mathbf{u}_N(t-\Delta t,\cdot)$ denotes the time-shift.

(B) There exists a universal constant $C > 0$ such that

$$\|P_N^{n+1} \frac{\mathbf{u}_N^{n+1} - \mathbf{u}_N^n}{\Delta t}\|_{(Q_N^{n+1})'} \leq C(\|\mathbf{u}_N^n\|_{V_N^n} + 1),\ n = 0,\dots,N-1,$$

 where P_N^{n+1} is the orthogonal projector onto $\overline{Q_N^{n+1}}^H$.

(C) The function spaces Q_N^n and V_N^n depend smoothly on time in the following sense:

 (C1) For every $N \in \mathbb{N}$, and for every $l \in \{1,\dots,N\}$ and $n \in \{1,\dots,N-l\}$, there exists a space $Q_N^{n,l} \subset V$ and the operators $J^i_{N,l,n} : Q_N^{n,l} \to Q_N^{n+i}$, $i = 0,\dots,l$, such that for every $\mathbf{q} \in Q_N^{n,l}$

$$\|J^i_{N,l,n}\mathbf{q}\|_{Q_N^{n+i}} \leq C\|\mathbf{q}\|_{Q_N^{n,l}}, \quad i \in \{0,\dots,l\}, \tag{3.50}$$

and

$$\left((J^{j+1}_{N,l,n}\mathbf{q} - J^{j}_{N,l,n}\mathbf{q}), \mathbf{u}^{n+j+1}_N\right)_H \le C\Delta t\|\mathbf{q}\|_{Q^{n,l}_N}\|\mathbf{u}^{n+j+1}_N\|_{V^{n+j+1}_N}, \quad j \in \{0,\dots,l-1\}, \tag{3.51}$$

$$\|J^i_{N,l,n}\mathbf{q} - \mathbf{q}\|_H \le C\sqrt{l\Delta t}\|\mathbf{q}\|_{Q^{n,l}_N}, \quad i \in \{0,\dots,l\}, \tag{3.52}$$

where $C > 0$ is independent of N, n and l.

(C2) Let $V^{n,l}_N = \overline{Q^{n,l}_N}^V$. There exist the functions $I^i_{N,l,n} : V^{n+1}_N \to V^{n,l}_N$, $i = 0,\dots,l$, and a universal constant $C > 0$ such that for every $\mathbf{v} \in V^{n+i}_N$

$$\|I^i_{N,l,n}\mathbf{v}\|_{V^{n,l}_N} \le C\|\mathbf{v}\|_{V^{n+i}_N}, \quad i \in \{0,\dots,l\}, \tag{3.53}$$

$$\|I^i_{N,l,n}\mathbf{v} - \mathbf{v}\|_H \le g(l\Delta t)\|\mathbf{v}\|_{V^{n+i}_N}, \quad i \in \{0,\dots,l\}, \tag{3.54}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a universal, monotonically increasing function such that $g(h) \to 0$ as $h \to 0$.

(C3) Uniform Ehrling property. For every $\delta > 0$ there exists a constant $C(\delta) > 0$ independent of N, l and n such that

$$\|\mathbf{v}\|_H \le \delta\|\mathbf{v}\|_{V^{n,l}_N} + C(\delta)\|\mathbf{v}\|_{(Q^{n,l}_N)'}. \tag{3.55}$$

then $\{\mathbf{u}_N\}$ is relatively compact in $L^2(0,T;H)$.

We want to apply Theorem 3.14 to our fluid-mesh-shell interaction problem. To state the compactness theorem we introduce the following overarching function spaces:

$$\begin{aligned} V &= H^s(\Omega_M) \times H^s(\omega) \times L^2(\mathcal{N}) \times L^2(\mathcal{N}), \\ H &= L^2(\Omega_M) \times L^2(\omega) \times H^{-s}(\mathcal{N}) \times H^{-s}(\mathcal{N}), \quad 0 < s < 1/2, \end{aligned}$$

where Ω_M is the maximal fluid domain containing all the time-dependent domains. Furthermore, for $n = 1,\dots,N$, we define the approximation solution spaces:

$$V^n_N = \{(\mathbf{u},\mathbf{v},\mathbf{k},\mathbf{z}) \in V^n_F \times H^{1/2}(\omega) \times L^2(\mathcal{N}) \times L^2(\mathcal{N}) : (\mathbf{u}\circ\boldsymbol{\phi}^n)|_\Gamma \circ \boldsymbol{\varphi} = \mathbf{v}\}, \tag{3.56}$$

and the approximation test spaces:

$$Q^n_N = \{(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta}) \in (V^n_F \cap H^5(\Omega^n)) \times V_K \times V_S \times V_S : (\boldsymbol{\upsilon}\circ\boldsymbol{\phi}^n)|_\Gamma \circ \boldsymbol{\varphi} = \boldsymbol{\psi}, \boldsymbol{\psi}\circ\boldsymbol{\pi} = \boldsymbol{\xi}\}. \tag{3.57}$$

Remark We assume that all the functions are extended by 0 to Ω_M.

It is also useful to write the weak formulation of the coupled, semi-discretized problem which we obtain by summing up the weak formulation for the semi-discretized structure subproblem (3.34) and the weak formulation for the semi-discretized fluid subproblem (3.40):

$$\begin{aligned}
&\rho_F \int_{\Omega^{n+1}} \frac{\mathbf{u}^{n+1}-\hat{\mathbf{u}}^n}{\Delta t}\cdot\boldsymbol{v} + \frac{\rho_F}{2}\int_{\Omega^{n+1}} (\nabla\cdot\mathbf{s}^{n+1,n})(\hat{\mathbf{u}}^n\cdot\boldsymbol{v})\\
&+\frac{\rho_F}{2}\int_{\Omega^{n+1}}\Big[((\hat{\mathbf{u}}^n-\mathbf{s}^{n+1,n})\cdot\nabla)\mathbf{u}^{n+1}\cdot\boldsymbol{v} - ((\hat{\mathbf{u}}^n-\mathbf{s}^{n+1,n})\cdot\nabla)\boldsymbol{v}\cdot\mathbf{u}^{n+1}\Big]\\
&+2\mu_F\int_{\Omega^{n+1}}\mathbf{D}(\mathbf{u}^{n+1}):\mathbf{D}(\boldsymbol{v}) + \rho_K h\int_\omega \frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\Delta t}\cdot\boldsymbol{\psi}R + a_K(\boldsymbol{\eta}^{n+1},\boldsymbol{\psi})\\
&+\rho_S\sum_{i=1}^{n_E} A_i\int_0^{l_i}\frac{\mathbf{k}_i^{n+1}-\mathbf{k}_i^n}{\Delta t}\cdot\boldsymbol{\xi}_i + \rho_S\sum_{i=1}^{n_E}\int_0^{l_i} M_i\frac{\mathbf{z}_i^{n+1}-\mathbf{z}_i^n}{\Delta t}\cdot\boldsymbol{\zeta}_i + a_S(\mathbf{w}^{n+1},\boldsymbol{\zeta})\\
&= P_{in}^n\int_{\Gamma_{in}} v_z - P_{out}^n\int_{\Gamma_{out}} v_z.
\end{aligned}\tag{3.58}$$

Theorem 3.15 Let $\{(\mathbf{u}_N,\mathbf{v}_N,\mathbf{k}_N,\mathbf{z}_N)\}$ be the sequence of approximate solutions defined in (3.43), satisfying the weak formulation (3.58) and uniform energy estimates from Theorem 3.6. Then $\{(\mathbf{u}_N,\mathbf{v}_N,\mathbf{k}_N,\mathbf{z}_N)\}$ is relatively compact in $L^2(0,T;H)$.

Remark For the simplicity of notation, throughout the rest of this section we will be assuming, without loss of generality, that all physical constants are equal 1.

Proof. We would like to show that the assumptions (A)-(C) from Theorem 3.14 hold.

Property A. We need to show that there exists a universal constant $C>0$ such that for every N, the estimates (A1)-(A3) hold. We start by showing (A1).

(A1) The $L^2(0,T;V_N^n)$ estimate:

$$\sum_{n=1}^N \|(\mathbf{u}_N^n,\mathbf{v}_N^n,\mathbf{k}_N^n,\mathbf{z}_N^n)\|^2_{V_N^n} = \sum_{n=1}^N\left(\|\mathbf{u}_N^n\|^2_{H^1(\Omega^n)}+\|\mathbf{v}_N^n\|^2_{H^{1/2}(\omega)}+\|\mathbf{k}_N^n\|^2_{L^2(\mathcal{N})}+\|\mathbf{z}_N^n\|^2_{L^2(\mathcal{N})}\right).$$

The approximate fluid and mesh velocities are bounded by Statement 1 from Theorem 3.6, i.e. uniform energy estimates. For the shell velocity, by trace theorem, we have

$$\|\mathbf{v}_N^n\|^2_{H^{1/2}(\omega)} \le C\|\mathbf{u}_N^n\|^2_{H^1(\Omega^n)},\tag{3.59}$$

and the right-hand side is again bounded by Theorem 3.6.

(A2) The $L^\infty(0,T;H)$ estimate:

$$\begin{aligned}
&\|(\mathbf{u}_N,\mathbf{v}_N,\mathbf{k}_N,\mathbf{z}_N)\|_{L^\infty(0,T;H)}\\
&=\|\mathbf{u}_N\|_{L^\infty(0,T;L^2(\Omega_M))}+\|\mathbf{v}_N\|_{L^\infty(0,T;L^2(\omega))}+\|\mathbf{k}_N\|_{L^\infty(0,T;H^{-s}(\mathcal{N}))}+\|\mathbf{z}_N\|_{L^\infty(0,T;H^{-s}(\mathcal{N}))}\\
&=\max_{n=1,\dots,N}\left(\|\mathbf{u}_N^n\|_{L^2(\Omega^n)}+\|\mathbf{v}_N^n\|_{L^2(\omega)}+\|\mathbf{k}_N^n\|_{H^{-s}(\mathcal{N})}+\|\mathbf{z}_N^n\|_{H^{-s}(\mathcal{N})}\right)\\
&\le\max_{n=1,\dots,N}\left(\|\mathbf{u}_N^n\|_{L^2(\Omega^n)}+\|\mathbf{v}_N^n\|_{L^2(\omega)}+\|\mathbf{k}_N^n\|_{L^2(\mathcal{N})}+\|\mathbf{z}_N^n\|_{L^2(\mathcal{N})}\right).
\end{aligned}$$

The right-hand side is bounded by Statement 1 from Theorem 3.6.

This completes the proof of Property A since the condition (A3) is a consequence of Property B which we show next (Theorem 3.2. in [62]).

Property B. We need to show uniform time-derivative bound, i.e. we need to estimate the following norm

$$\begin{aligned}
&\left\| P_N^{n+1} \frac{(\mathbf{u}_N^{n+1}, \mathbf{v}_N^{n+1}, \mathbf{k}_N^{n+1}, \mathbf{z}_N^{n+1}) - (\mathbf{u}_N^n, \mathbf{v}_N^n, \mathbf{k}_N^n, \mathbf{z}_N^n)}{\Delta t} \right\|_{(Q_N^{n+1})'} \\
&= \sup_{\|(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\|_{Q_N^{n+1}}=1} \left| \int_{\Omega^{n+1}} \frac{\mathbf{u}_N^{n+1} - \mathbf{u}_N^n}{\Delta t} \cdot \boldsymbol{\upsilon} + \int_\omega \frac{\mathbf{v}_N^{n+1} - \mathbf{v}_N^n}{\Delta t} \cdot \boldsymbol{\psi} \right. \\
&\quad \left. + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{k}_N^{n+1})_i - (\mathbf{k}_N^n)_i}{\Delta t} \cdot \boldsymbol{\xi}_i + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{z}_N^{n+1})_i - (\mathbf{z}_N^n)_i}{\Delta t} \cdot \boldsymbol{\zeta}_i \right|.
\end{aligned}$$

We start by adding and subtracting the function $\hat{\mathbf{u}}_N^n$ which is defined in (3.39):

$$\begin{aligned}
&\left| \int_{\Omega^{n+1}} \frac{\mathbf{u}_N^{n+1} - \mathbf{u}_N^n \pm \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \boldsymbol{\upsilon} + \int_\omega \frac{\mathbf{v}_N^{n+1} - \mathbf{v}_N^n}{\Delta t} \cdot \boldsymbol{\psi} \right. \\
&\quad \left. + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{k}_N^{n+1})_i - (\mathbf{k}_N^n)_i}{\Delta t} \cdot \boldsymbol{\xi}_i + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{z}_N^{n+1})_i - (\mathbf{z}_N^n)_i}{\Delta t} \cdot \boldsymbol{\zeta}_i \right| \\
&\le \left| \int_{\Omega^{n+1}} \frac{\mathbf{u}_N^{n+1} - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \boldsymbol{\upsilon} + \int_\omega \frac{\mathbf{v}_N^{n+1} - \mathbf{v}_N^n}{\Delta t} \cdot \boldsymbol{\psi} \right. \\
&\quad \left. + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{k}_N^{n+1})_i - (\mathbf{k}_N^n)_i}{\Delta t} \cdot \boldsymbol{\xi}_i + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{z}_N^{n+1})_i - (\mathbf{z}_N^n)_i}{\Delta t} \cdot \boldsymbol{\zeta}_i \right| + \left| \int_{\Omega^{n+1}} \frac{\hat{\mathbf{u}}_N^n - \mathbf{u}_N^n}{\Delta t} \cdot \boldsymbol{\upsilon} \right|.
\end{aligned}$$

We rewrite the first term by using the weak formulation (3.58) to obtain the following estimate:

$$\begin{aligned}
&\left| \int_{\Omega^{n+1}} \frac{\mathbf{u}_N^{n+1} - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \boldsymbol{\upsilon} + \int_\omega \frac{\mathbf{v}_N^{n+1} - \mathbf{v}_N^n}{\Delta t} \cdot \boldsymbol{\psi} \right. \\
&\quad \left. + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{k}_N^{n+1})_i - (\mathbf{k}_N^n)_i}{\Delta t} \cdot \boldsymbol{\xi}_i + \sum_{i=1}^{n_E} \int_0^{l_i} \frac{(\mathbf{z}_N^{n+1})_i - (\mathbf{z}_N^n)_i}{\Delta t} \cdot \boldsymbol{\zeta}_i \right| \\
&\le C_1 \|\mathbf{v}_N^{n+1/2}\|_{L^2} \|\hat{\mathbf{u}}_N^n\|_{L^2} \|\boldsymbol{\upsilon}\|_{L^\infty} + C_2 \|\nabla \mathbf{u}_N^{n+1}\|_{L^2} \|\hat{\mathbf{u}}_N^n\|_{L^2} \|\|\boldsymbol{\upsilon}\|_{L^\infty} + C_3 \|\nabla \mathbf{u}_N^{n+1}\|_{L^2} \|\nabla \boldsymbol{\upsilon}\|_{L^2} \\
&\quad + C_4 \|\boldsymbol{\eta}\|_{H^2} \|\boldsymbol{\psi}\|_{H^2} + C_5 \|\partial_s \mathbf{w}\|_{L^2} \|\partial_s \boldsymbol{\zeta}\|_{L^2} + C_6 \|\boldsymbol{\upsilon}\|_{L^\infty} \\
&\le C(\|(\mathbf{u}_N^{n+1}, \mathbf{v}_N^{n+1}, \mathbf{k}_N^{n+1}, , \mathbf{k}_N^{n+1})\|_{V_N^{n+1}} + 1) \|(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})\|_{Q_N^{n+1}}.
\end{aligned}$$

To estimate the second term, we first notice that function $\hat{\mathbf{u}}_N^n$ is 0 outside domain Ω^{n+1}, while function $\mathbf{u}_N^n$ is 0 outside domain Ω^n. This is why we introduce $A = \Omega^{n+1} \cap \Omega^n, B_1 = \Omega^{n+1} \backslash \Omega^n$ and $B_2 = \Omega^n \backslash \Omega^{n+1}$, and estimate the integrals over A, B_1 and B_2 separately.

First we start with the integral over A, i.e. over the area of the intersection of two

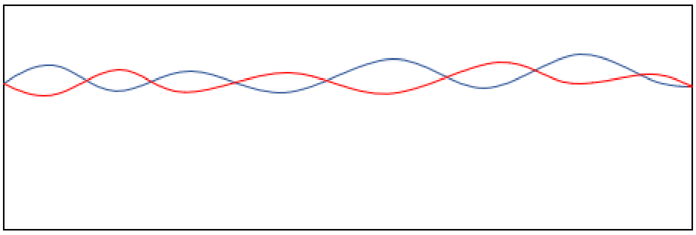

Figure 3.2: The 2d fluid domains at time steps $t = n\Delta t$ and $t = (n+1)\Delta t$

consecutive domains Ω^n and Ω^{n+1} :

$$\begin{aligned}
\left|\int_A (\hat{\mathbf{u}}_N^n - \mathbf{u}_N^n)\cdot \boldsymbol{v}\right| &= \left|\int_A (\mathbf{u}_N^n \circ A^{n+1,n} - \mathbf{u}_N^n)\cdot \boldsymbol{v}\right| \\
&= \left|\int_A \left(\mathbf{u}_N^n(z, \frac{R+\tilde{\eta}_N^n(z,\theta)}{R+\tilde{\eta}_N^{n+1}(z,\theta)} r, \theta) - \mathbf{u}_N^n(z,r,\theta)\right)\cdot \boldsymbol{v}(z,r,\theta)\ dzdrd\theta\right|
\end{aligned}$$

By using the mean value theorem and Hölder's inequality we get:

$$\begin{aligned}
\left|\int_A (\hat{\mathbf{u}}_N^n - \mathbf{u}_N^n)\cdot \boldsymbol{v}\right| &\le C\|\nabla \mathbf{u}_N^n \cdot (\tilde{\eta}_N^n - \tilde{\eta}_N^{n+1})\mathbf{e}_r\|_{L^1(A)}\|\boldsymbol{v}\|_{L^\infty(A)} \\
&\le C\Delta t\|\nabla \mathbf{u}_N^n\|_{L^2(A)}\|\mathbf{v}_N^{n+1/2}\|_{L^2(A)}\|\boldsymbol{v}\|_{L^\infty(A)} \\
&\le C\|\mathbf{u}_N^n\|_{H^1(A)}\|\boldsymbol{v}\|_{H^5(A)} \\
&\le C\|(\mathbf{u}_N^n, \mathbf{v}_N^n, \mathbf{k}_N^n, \mathbf{z}_N^n)\|_{V_N^n}\|(\boldsymbol{v},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\|_{Q_N^n}.
\end{aligned}$$

Notice how the higher regularity of the test space Q_N^n provided the upper bound for the L^∞-norm of the test function $\boldsymbol{v}$.

To estimate the integral over B_1, we use the fact that $\mathbf{u}_N^n = 0$ on B_1 to obtain:

$$\begin{aligned}
\left|\int_{B_1} (\hat{\mathbf{u}}_N^n - \mathbf{u}_N^n)\cdot \boldsymbol{v}\right| &= \left|\int_{B_1} \hat{\mathbf{u}}_N^n(z,r,\theta)\cdot \boldsymbol{v}(z,r,\theta)\ dzdrd\theta\right| \\
&= \left|\int_\omega \left(\int_{R+\tilde{\eta}_N^n}^{R+\tilde{\eta}_N^{n+1}} \hat{\mathbf{u}}_N^n(z,r,\theta)\cdot \boldsymbol{v}(z,r,\theta)\ dr\right) dzd\theta\right| \\
&\le \left|\int_\omega \max_r(\hat{\mathbf{u}}_N^n(z,r,\theta)\cdot \boldsymbol{v}(z,r,\theta)) \int_{R+\tilde{\eta}_N^n}^{R+\tilde{\eta}_N^{n+1}} drdzd\theta\right| \\
&\le C\int_\omega \|\partial_r u_r^n(z,\cdot,\theta)\|_{L^2_r}\|\boldsymbol{v}\|_{L^\infty}\|\Delta t \mathbf{v}_N^{n+1/2}\|_{L^2}\ dzd\theta \\
&\le C\Delta t\|\nabla \mathbf{u}_N^n\|_{L^2}\|\boldsymbol{v}\|_{L^\infty} \\
&\le C\|(\mathbf{u}_N^n, \mathbf{v}_N^n, \mathbf{k}_N^n, \mathbf{z}_N^n)\|_{V_N^n}\|(\boldsymbol{v},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\|_{Q_N^n}.
\end{aligned}$$

The integral over B_2 can be estimated in the same way as the integral over B_1 by using the fact that $\hat{\mathbf{u}}_N^n = 0$ on B_2. These estimates, together with the estimate obtained from the weak formulation, complete the proof of Property B, i.e. we have

$$\left\| P_N^{n+1} \frac{(\mathbf{u}_N^{n+1}, \mathbf{v}_N^{n+1}, \mathbf{k}_N^{n+1}, \mathbf{z}_N^{n+1}) - (\mathbf{u}_N^n, \mathbf{v}_N^n, \mathbf{k}_N^n, \mathbf{z}_N^n)}{\Delta t} \right\|_{(Q_N^{n+1})'} \leq C \left(\|(\mathbf{u}_N^n, \mathbf{v}_N^n, \mathbf{k}_N^n, \mathbf{z}_N^n)\|_{V_N^n} + 1 \right).$$

Property C. We need to show the smooth dependence of function spaces on time. Before defining the common test space required by Property C1, we first define the "local" maximal domain $\Omega^{n,l}$ which contains all the fluid domains $\Omega^{n+i}, i = 0, \ldots, l$:

$$\Omega^{n,l} = \{(z, r, \theta) : z \in (0, L), r \leq R + \tilde{\eta}^{n,l}(z, \theta), \theta \in (0, 2\pi)\}, \tag{3.60}$$

where $\tilde{\eta}^{n,l}(z, \theta) = \max_{i=0,\ldots,l} \tilde{\eta}^{n+i}(z, \theta)$, mollified if necessary to get the smooth functions.

Property C1. A common test space is then defined in the following way:

$$Q_N^{n,l} = \{(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in (V_F(\Omega^{n,l}) \cap H^5(\Omega^{n,l})) \times V_K \times V_S \times V_S : (\boldsymbol{\upsilon} \circ \boldsymbol{\phi}^n)|_\Gamma \circ \boldsymbol{\varphi} = \boldsymbol{\psi}, \boldsymbol{\psi} \circ \boldsymbol{\pi} = \boldsymbol{\xi}\}. \tag{3.61}$$

Let us take $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in Q_N^{n,l}$. We define $J_{N,l,n}^i$ as a restriction:

$$J_{N,l,n}^i(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) = (\boldsymbol{\upsilon}|_{\Omega^{n+i}}, \boldsymbol{\upsilon}|_{\Gamma^{n+i}} \circ \boldsymbol{\varphi}, (\boldsymbol{\upsilon}|_{\Gamma^{n+i}} \circ \boldsymbol{\varphi})|_{\mathcal{N}}, \boldsymbol{\zeta})$$

and set

$$(\boldsymbol{\upsilon}^i, \boldsymbol{\psi}^i, \boldsymbol{\xi}^i, \boldsymbol{\zeta}^i) := (\boldsymbol{\upsilon}|_{\Omega^{n+i}}, \boldsymbol{\upsilon}|_{\Gamma^{n+i}} \circ \boldsymbol{\varphi}, (\boldsymbol{\upsilon}|_{\Gamma^{n+i}} \circ \boldsymbol{\varphi})|_{\mathcal{N}}, \boldsymbol{\zeta}). \tag{3.62}$$

Property (3.50) follows from the definition of the mapping $J_{N,l,n}^i$. To verify property (3.51), we need to calculate

$$\left(J^{j+1}(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) - J^j(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}), (\mathbf{u}_N^{n+j+1}, \mathbf{v}_N^{n+j+1}, \mathbf{k}_N^{n+j+1}, \mathbf{z}_N^{n+j+1}) \right)_H. \tag{3.63}$$

We estimate each term separately. We estimate the first term in the similar way as we did in Property B:

$$\begin{aligned}
\left| \int_{\Omega_M} (\boldsymbol{\upsilon}^{j+1} - \boldsymbol{\upsilon}^j) \cdot \mathbf{u}_N^{n+j+1} \right| &= \left| \int_{\Omega^{n+j+1} \Delta \Omega^{n+j}} \boldsymbol{\upsilon} \cdot \mathbf{u}_N^{n+j+1} \right| \\
&= \left| \int_\omega \left(\int_{R+\tilde{\eta}_N^{n+j}}^{R+\tilde{\eta}_N^{n+j+1}} \boldsymbol{\upsilon}(z, r, \theta) \cdot \mathbf{u}_N^{n+j+1}(z, r, \theta) \, dr \right) dz d\theta \right| \\
&\leq \left| \int_\omega \max_r (\boldsymbol{\upsilon}(z, r, \theta) \cdot \mathbf{u}_N^{n+j+1}(z, r, \theta)) \int_{R+\tilde{\eta}_N^{n+j}}^{R+\tilde{\eta}_N^{n+j+1}} dr dz d\theta \right| \\
&\leq C \int_\omega \|\boldsymbol{\upsilon}\|_{L^\infty} \|\partial_r u_r^{n+j+1}(z, \cdot, \theta)\|_{L_r^2} \|\Delta t \mathbf{v}_N^{n+j+1/2}\|_{L^2} \, dz d\theta \\
&\leq C \|\boldsymbol{\upsilon}\|_{L^\infty} \|\nabla \mathbf{u}_N^{n+j+1}\|_{L^2} \|\Delta t \mathbf{v}_N^{n+j+1/2}\|_{L^2} \\
&\leq C \Delta t \|(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})\|_{Q_N^{n,l}} \|(\mathbf{u}_N^{n+j+1}, \mathbf{v}_N^{n+j+1}, \mathbf{k}_N^{n+j+1}, \mathbf{z}_N^{n+j+1})\|_{V_N^{n+j+1}}.
\end{aligned}$$

Before estimating the second term, note that

$$\boldsymbol{\psi}^i = \boldsymbol{\upsilon}|_{\Gamma^{n+i}} \circ \boldsymbol{\varphi} = (\boldsymbol{\upsilon} \circ \boldsymbol{\phi}^{n+i})|_\Gamma \circ \boldsymbol{\varphi},$$

so, by using the mean value theorem, we get

$$\begin{aligned} \left|\int_\omega (\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j) \cdot \mathbf{v}_N^{n+j+1}\right| &\leq \int_\omega |(\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j) \cdot \mathbf{v}_N^{n+j+1}| \leq \|\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j\|_{L^2(\omega)} \|\mathbf{v}_N^{n+j+1}\|_{L^2(\omega)} \\ &= \|(\boldsymbol{\upsilon}(\boldsymbol{\phi}^{n+j+1}(\cdot)) - \boldsymbol{\upsilon}(\boldsymbol{\phi}^{n+j}(\cdot)))|_\Gamma \circ \boldsymbol{\varphi}\|_{L^2(\omega)} \|\mathbf{v}_N^{n+j+1}\|_{L^2(\omega)} \\ &\leq \|\nabla \boldsymbol{\upsilon}\|_{L^\infty(\Omega^{n,l})} \|(\boldsymbol{\phi}^{n+j+1} - \boldsymbol{\phi}^{n+j})|_\Gamma \circ \boldsymbol{\varphi}\|_{L^2(\omega)} \|\mathbf{v}_N^{n+j+1}\|_{L^2(\omega)}. \end{aligned} \tag{3.64}$$

Recall that $\boldsymbol{\phi}^i|_\Gamma \circ \boldsymbol{\varphi} = \mathbf{id} + \boldsymbol{\eta}_N^i$, so we can further estimate the right-hand side to obtain

$$\begin{aligned} \left|\int_\omega (\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j) \cdot \mathbf{v}_N^{n+j+1}\right| &\leq \|\nabla \boldsymbol{\upsilon}\|_{L^\infty(\Omega^{n,l})} \|\boldsymbol{\eta}_N^{n+j+1} - \boldsymbol{\eta}_N^{n+j}\|_{L^2(\omega)} \|\mathbf{v}_N^{n+j+1}\|_{L^2(\omega)} \\ &= \|\nabla \boldsymbol{\upsilon}\|_{L^\infty(\Omega^{n,l})} \Delta t \|\mathbf{v}_N^{n+j+1/2}\|_{L^2(\omega)} \|\mathbf{v}_N^{n+j+1}\|_{L^2(\omega)} \\ &\leq C\Delta t \|(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})\|_{Q_N^{n,l}} \|(\mathbf{u}_N^{n+j+1}, \mathbf{v}_N^{n+j+1}, \mathbf{k}_N^{n+j+1}, \mathbf{z}_N^{n+j+1})\|_{V_N^{n+j+1}}. \end{aligned} \tag{3.65}$$

What is left is to take care of the term $\left|\sum_{n=1}^{n_E} \int_0^{l_i} (\boldsymbol{\xi}_i^{j+1} - \boldsymbol{\xi}_i^j) \cdot \mathbf{k}_i^{n+j+1}\right|$. Recall that

$$\boldsymbol{\xi}^i = (\boldsymbol{\upsilon}|_{\Gamma^{n+i}} \circ \boldsymbol{\varphi})|_\mathcal{N} = \boldsymbol{\psi}^i|_\mathcal{N} = \boldsymbol{\psi}^i \circ \boldsymbol{\pi},$$

so

$$\begin{aligned} \left|\sum_{n=1}^{n_E} \int_0^{l_i} (\boldsymbol{\xi}_i^{j+1} - \boldsymbol{\xi}_i^j) \cdot \mathbf{k}_i^{n+j+1}\right| &\leq \|\boldsymbol{\xi}^{j+1} - \boldsymbol{\xi}^j\|_{L^2(\mathcal{N})} \|\mathbf{k}_N^{n+j+1}\|_{L^2(\mathcal{N})} \\ &= \|(\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j)|_\mathcal{N}\|_{L^2(\mathcal{N})} \|\mathbf{k}_N^{n+j+1}\|_{L^2(\mathcal{N})} \\ &\leq C \|\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j\|_{L^2(\omega)} \|\mathbf{k}_N^{n+j+1}\|_{L^2(\mathcal{N})} \\ &\leq C\Delta t \|(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})\|_{Q_N^{n,l}} \|(\mathbf{u}_N^{n+j+1}, \mathbf{v}_N^{n+j+1}, \mathbf{k}_N^{n+j+1}, \mathbf{z}_N^{n+j+1})\|_{V_N^{n+j+1}}, \end{aligned}$$

where in the last inequality we used the fact that $\|\boldsymbol{\psi}^{j+1} - \boldsymbol{\psi}^j\|_{L^2(\omega)}$ is bounded from (3.64) and (3.65). At last, we have to check that property (3.52) is valid, i.e.

$$\|J_{N,l,n}^i(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) - (\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})\|_H \leq C\sqrt{l\Delta t} \|(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})\|_{Q_N^{n,l}}.$$

It is clear that $J_{N,l,n}^i(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ and $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ differ only in the region $\Omega^{n,l} \backslash \Omega^{n+i}$, so the H-norm of the difference between the two functions can be bounded by the $Q_N^{n,l}$-norm of $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ and the H^1-norm of the difference $\tilde{\boldsymbol{\eta}}_N^{n,l} - \tilde{\boldsymbol{\eta}}_N^{n+i}$, $i = 0, \dots, l$. Since $\tilde{\boldsymbol{\eta}}_N^{n,l}$ is the maximum of the finitely many functions $\tilde{\boldsymbol{\eta}}_N^{n+i}$, we calculate the H^1-norm of the difference

$\tilde{\boldsymbol{\eta}}_N^{n+i} - \tilde{\boldsymbol{\eta}}_N^n, i = 1, \dots, l$, and get, the same way as in (3.47), that the upper bound on $\|\tilde{\boldsymbol{\eta}}_N^{n+i} - \tilde{\boldsymbol{\eta}}_N^n\|_{H^1(\omega)}$ only depends on the width of the time interval, which is $l\Delta t$, namely

$$\|\tilde{\boldsymbol{\eta}}_N^{n+i} - \tilde{\boldsymbol{\eta}}_N^n\|_{H^1(\omega)} \leq C\sqrt{l\Delta t},\ i = 1, \dots, l.$$

This completes the verification of Property C1.

Property C2. This property requires us to check approximation properties of solution spaces. More precisely, we need to verify that every function in $V_N^{n+i}, i = 0, \dots, l$, defined by the time-shift $i\Delta t$, can be approximated by a function in the common solution space, which we will denote by $V_N^{n,l}$. To do so, we need to construct a mapping $I_{N,l,n}^i : V_N^{n+i} \to V_N^{n,l}$ with good approximation properties (3.53) and (3.54). We start by defining the common solution space $V_N^{n,l}$ to be the closure of $Q_N^{n,l}$ in V (for $s = 1/2$):

$$\begin{aligned} V_N^{n,l} = \{(\mathbf{u}, \mathbf{v}, \mathbf{k}, \mathbf{z}) \in H^{1/2}(\Omega^{n,l}) \times H^{1/2}(\omega) \times L^2(\mathcal{N}) \times L^2(\mathcal{N}) : \nabla \cdot \mathbf{u} = 0, \\ ((\mathbf{u} \circ \boldsymbol{\phi}^n)|_\Gamma \circ \boldsymbol{\varphi} - \mathbf{v}) \cdot \mathbf{n} = 0\}. \end{aligned} \tag{3.66}$$

We want to construct a divergence free extension $\tilde{\mathbf{u}}_N^n$ of the function $\mathbf{u}_N^n \in V_F^n = V_F(\Omega^n)$ to the maximal domain Ω_M. This is done in the following lemma:

Lemma 3.16 Let $\mathbf{u} \in V_F^n$. Then there exist a divergence free function $\overline{\mathbf{u}} \in V$ such that $\overline{\mathbf{u}}|_{\Omega^n} = \mathbf{u}$ and

$$\|\overline{\mathbf{u}}\|_V \leq C\|\mathbf{u}\|_{V_F^n}, \tag{3.67}$$

where C is independent of N and n.

Proof. Take $\mathbf{u} \in V_F^n$ and set

$$\mathbf{u}_R = \mathbf{u} \circ \boldsymbol{\phi}^n. \tag{3.68}$$

Recall that $\boldsymbol{\phi}^n(\Omega) = \Omega^n$, so $\mathbf{u}_R$ is a function defined on the reference domain Ω. For $\mathbf{u}_R$ we have the following estimate:

$$\|\mathbf{u}_R\|_{H^1(\Omega)} \leq \|\mathbf{u}\|_{H^1(\Omega^n)}\|\boldsymbol{\phi}^n\|_{W^{1,\infty}(\Omega)} \leq C\|\mathbf{u}\|_{H^1(\Omega^n)}.$$

Now we extend $\mathbf{u}_R$ to $\mathbb{R}^3$ and then define $\tilde{\mathbf{u}}_R$ as a restriction of that particular extension to Ω_M. It is clear that the following bound holds:

$$\|\tilde{\mathbf{u}}_R\|_{H^1(\Omega_M)} \leq C\|\mathbf{u}\|_{H^1(\Omega^n)}.$$

Since we are constructing our divergence free extension from Ω^n to Ω_M, we use the mapping $\boldsymbol{\phi}^n$ to go back to the physical domain, i.e. we define the extended function $\tilde{\mathbf{u}}$ in the following way:

$$\tilde{\mathbf{u}} = \begin{cases} \tilde{\mathbf{u}}_R \circ (\boldsymbol{\phi}^n)^{-1} \text{ in } \Omega^n, \\ \tilde{\mathbf{u}}_R \circ (\tilde{\boldsymbol{\phi}}^n)^{-1} \text{ in } \Omega_M \backslash \Omega^n, \end{cases}$$

where $\tilde{\boldsymbol{\phi}}^n : \Omega_M \backslash \Omega \to \Omega_M \backslash \Omega^n$ is the extension of the mapping $\boldsymbol{\phi}^n = \boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}}(n\Delta t, \cdot)$ to the maximal domain Ω_M. One can easily check that the following bound holds:

$$\|\tilde{\mathbf{u}}\|_{H^1(\Omega_M)} \leq C\|\mathbf{u}\|_{H^1(\Omega^n)}.$$

Unfortunately, $\tilde{\mathbf{u}}$ is not divergence free so we need to "correct" it. We aim on applying Theorem III.3.1. from [43] which deals with the problem of finding a vector field $\mathbf{v} \in W^{1,p}(\Omega)$ which satisfies

$$\nabla \cdot \mathbf{v} = \mathbf{f}, \tag{3.69}$$

where $\mathbf{f} \in L^p(\Omega)$ such that $\int_\Omega \mathbf{f} = 0$. Of course, the solvability of the problem requires some regularity on Ω which we will specify later.

First, we need to find a mapping $\boldsymbol{\kappa} : (\Omega_M \backslash \Omega^n) \to \mathbb{R}^3$ such that the following conditions are satisfied:

(i) $\boldsymbol{\kappa}|_{\Gamma^n} = 0$,

(ii) $\displaystyle\int_{\partial(\Omega_M \backslash \Omega^n)} (\tilde{\mathbf{u}} + \boldsymbol{\kappa}) \cdot \mathbf{n} = 0$,

(iii) $\|\boldsymbol{\kappa}\|_{H^1(\Omega_M \backslash \Omega^n)} \leq C\|\mathbf{u}\|_{H^1(\Omega^n)}$.

The first condition will ensure that $(\tilde{\mathbf{u}} + \boldsymbol{\kappa})|_{\Gamma^n} = \mathbf{u}$, while the second condition is a compatibility condition corresponding to the fact that the integral of the right-hand side of problem (3.69) has to be zero.

We take function $\boldsymbol{\kappa}$ to be the function from $C_c^\infty(\overline{\omega} \times (R + \tilde{\eta}^n, R_{max}])$. It is clear that the first condition is then automatically satisfied. The second condition reads:

$$\int_{\partial\Omega_M} (\tilde{\mathbf{u}} + \boldsymbol{\kappa}) \cdot \mathbf{n} = 0,$$

and if we define $\boldsymbol{\kappa} := -c\boldsymbol{\iota}$, where $c = \int_{\partial\Omega_M} \tilde{\mathbf{u}} \cdot \mathbf{n}$ and $\boldsymbol{\iota}$ is such that $\int_{\partial\Omega_M} \boldsymbol{\iota} \cdot \mathbf{n} = 1$, then the second condition is also satisfied. If we choose $\boldsymbol{\iota}$ independent of $\tilde{\mathbf{u}}$ and $\mathbf{n}$ (for example, we take $\boldsymbol{\iota}$ such that $\operatorname{supp} \boldsymbol{\iota}$ does not intersect any of Ω^n), then we get the desired upper bound on the norm of $\boldsymbol{\kappa}$, i.e.

$$\|\boldsymbol{\kappa}\|_{H^1(\Omega_M \backslash \Omega^n)} \leq C\|\tilde{\mathbf{u}}\|_{H^1(\Omega_M)} \leq C\|\mathbf{u}\|_{H^1(\Omega^n)}.$$

Finally, since our approximate domain Ω^n is Lipschitz, the complementary domain $\Omega_M \backslash \Omega^n$ is also Lipschitz, so we can decompose it as a union of finitely many star-shaped domains. Let us briefly describe why we can choose the number of domains to be independent of n. More precisely, we claim that we can write the complementary domain $\Omega_M \backslash \Omega^n$ as a union of exactly 4 star-shaped domains (with respect to fixed balls), which we obtain by taking the longitudinal and cross-sectional cut of our domain. Since the approximate reparameterized shell displacement $\tilde{\boldsymbol{\eta}}_N$ satisfies $\|\tilde{\boldsymbol{\eta}}_N\|_{W^{1,\infty}(\omega)} \leq C$, i.e. Lipschitz norm of $\tilde{\boldsymbol{\eta}}_N$ is small, we know that $\tilde{\boldsymbol{\eta}}_N$ is limited in how fast it can change. For that reason, we can find a ball (in each of the four sections), such that the angle between a line connecting an arbitrary point from the ball with an arbitrary point from Γ^n (corresponding to that section) with the normal vector at the latter point will always be less than $\pi/2$, i.e. the line will never intersect Γ^n. This tells us that each of the four sections is a star-shaped domain with respect to a fixed ball.

We are now ready to apply Theorem III.3.1. from [43] with $\mathbf{f} = -\nabla \cdot (\tilde{\mathbf{u}} + \boldsymbol{\kappa})$ to see that there exists $\mathbf{V}$ such that the following holds:

$$\nabla \cdot \mathbf{V} = -\nabla \cdot (\tilde{\mathbf{u}} + \boldsymbol{\kappa})$$

with

$$\|\mathbf{V}\|_{H^1(\Omega_M)} \leq C\|\tilde{\mathbf{u}} + \boldsymbol{\kappa}\|_{H^1(\Omega_M)}.$$

Finally, if we define $\overline{\mathbf{u}} := \tilde{\mathbf{u}} + \boldsymbol{\kappa} + \mathbf{V}$, then $\overline{\mathbf{u}}$ is a divergence free extension of the function $\mathbf{u} \in V_F^n$ to the maximal domain Ω_M, that satisfies the desired estimate $\|\overline{\mathbf{u}}\|_V \leq C\|\mathbf{u}\|_{V_F^n}$. $\square$

Now we define mappings $I^i_{N,l,n} : V_N^{n+i} \to V_N^{n,l}$ in the following way:

$$I^i_{N,l,n}(\mathbf{u}_N^{n+i}, \mathbf{v}_N^{n+i}, \mathbf{k}_N^{n+i}, \mathbf{z}_N^{n+i}) = (\overline{\mathbf{u}}_N^{n+i}|_{\Omega^{n,l}}, (\overline{\mathbf{u}}_N^{n+i}|_{\Omega^{n,l}} \circ \boldsymbol{\phi}^n)|_\Gamma \circ \boldsymbol{\varphi}, \mathbf{k}_N^{n+i}, \mathbf{z}_N^{n+i}) \tag{3.70}$$

What is left is to prove that inequalities (3.53) and (3.54) hold. The inequality (3.53) follows from the definition of mappings $I^i_{N,l,n}$ and from Lemma 3.16. To see that inequality (3.54) holds, we need to prove that there exists a universal, monotonically increasing function g, which converges to 0 as $h \to 0$, where $h = l\Delta t$, such that

$$\begin{aligned} \|I^i_{N,l,n}(\mathbf{u}_N^{n+i}, \mathbf{v}_N^{n+i}, \mathbf{k}_N^{n+i}, \mathbf{z}_N^{n+i}) - (\mathbf{u}_N^{n+i}, \mathbf{v}_N^{n+i}, \mathbf{k}_N^{n+i}, \mathbf{z}_N^{n+i})\|_H \\ \leq g(l\Delta t)\|(\mathbf{u}_N^{n+i}, \mathbf{v}_N^{n+i}, \mathbf{k}_N^{n+i}, \mathbf{z}_N^{n+i})\|_{V_N^{n+i}}. \end{aligned}$$

To simplify notation, we drop the subscripts N, l, n, and take care of each term separately:

$$\begin{aligned} \|I^i \mathbf{u}_N^{n+i} - \mathbf{u}_N^{n+i}\|_{L^2(\Omega_M)} &= \|\overline{\mathbf{u}}_N^{n+i}|_{\Omega^{n,l}} - \mathbf{u}_N^{n+i}\|_{L^2(\Omega_M)} = \left(\int_{\Omega^{n,l}\backslash\Omega^{n+i}} |\overline{\mathbf{u}}_N^{n+i}|^2\right)^{1/2} \\ &\leq C\|\nabla \overline{\mathbf{u}}_N^{n+i}\|_{L^2(\Omega_M)} \left|\int_{R+\tilde{\eta}^{n+i}}^{R+\tilde{\eta}^{n,l}} dr\right|^{1/2} dzd\theta \leq C\sqrt{l\Delta t}\|\overline{\mathbf{u}}_N^{n+i}\|_{H^1(\Omega_M)} \\ &\leq C\sqrt{l\Delta t}\|\mathbf{u}_N^{n+i}\|_{V_F^{n+i}}. \end{aligned}$$

We estimate the second term by using the mean value theorem, just like in (3.64):

$$\begin{aligned}\|I^i\mathbf{v}_N^{n+i}-\mathbf{v}_N^{n+i}\|_{L^2(\omega)} &= \|(\overline{\mathbf{u}}_N^{n+i}|_{\Omega^{n,l}}\circ\boldsymbol{\phi}^n)|_\Gamma\circ\boldsymbol{\varphi}-(\mathbf{u}_N^{n+i}\circ\boldsymbol{\phi}^{n+i})|_\Gamma\circ\boldsymbol{\varphi}\|_{L^2(\omega)}\\ &\le C\|\nabla\overline{\mathbf{u}}_N^{n+i}\|_{L^2(\Omega_M)}\|\tilde{\boldsymbol{\eta}}_N^{n+i}-\tilde{\boldsymbol{\eta}}_N^{n}\|_{L^\infty(\omega)}\\ &\le C\sqrt{l\Delta t}\|\mathbf{u}_N^{n+i}\|_{V_F^{n+i}}.\end{aligned}$$

Taking into account the previous estimates, we get

$$\begin{aligned}&\|I^i_{N,l,n}(\mathbf{u}_N^{n+i},\mathbf{v}_N^{n+i},\mathbf{k}_N^{n+i},\mathbf{z}_N^{n+i})-(\mathbf{u}_N^{n+i},\mathbf{v}_N^{n+i},\mathbf{k}_N^{n+i},\mathbf{z}_N^{n+i})\|_H\\ &=\|I^i\mathbf{u}_N^{n+i}-\mathbf{u}_N^{n+i}\|_{L^2(\Omega_M)}+\|I^i\mathbf{v}_N^{n+i}-\mathbf{v}_N^{n+i}\|_{L^2(\omega)}\\ &\le C\sqrt{l\Delta t}\|\mathbf{u}_N^{n+i}\|_{V_F^{n+i}}\le g(l\Delta t)\|(\mathbf{u}_N^{n+i},\mathbf{v}_N^{n+i},\mathbf{k}_N^{n+i},\mathbf{z}_N^{n+i})\|_{V_N^{n+i}}.\end{aligned}$$

This finishes the proof of Property C2.

Property C3. We need to prove the uniform Ehrling property, stated in (3.55). The main difficulty comes from the fact that we have to work with moving domains, which are parameterized by N, l, n. To show that the uniform Ehrling estimate holds, independently of all three parameters, we simplify the notation, and replace the indices N, l, n with only one index n, so that our function spaces are now denoted V_n, H_n and Q_n'. We show the uniform Ehrling property by contradiction. We start by assuming that the statement of the uniform Ehrling property (3.55) is false. More precisely, we assume that there exists a $\delta_0 > 0$ and a sequence $\mathbf{h}_n = (\mathbf{u}_n, \mathbf{v}_n, \mathbf{k}_n, \mathbf{z}_n) \in H_n$ such that

$$\|\mathbf{h}_n\|_H = \|\mathbf{h}_n\|_{H^n} > \delta_0\|\mathbf{h}\|_{V_n} + n\|\mathbf{h}\|_{Q_n'}.$$

Here we have extended the functions $\mathbf{u}_n$ onto the maximal domain Ω_M by 0. We also replace the V_n norm on the right-hand side by the norm on V :

$$\|\mathbf{h}_n\|_H > \delta_0\|\mathbf{h}\|_{V_n} + n\|\mathbf{h}\|_{Q_n'} \ge C\delta_0\|\mathbf{h}_n\|_V + n\|\mathbf{h}_n\|_{Q_n'}.$$

Without the loss of generality we can assume that our sequence $(\mathbf{h}_n)$ is such that $\|\mathbf{h}_n\|_H = 1$. The two terms on the right-hand side are uniformly bounded in n, which implies that there exists a subsequence, which we again denote by $(\mathbf{h}_n)$, such that:

$$\|\mathbf{h}_n\|_H = 1, \quad \|\mathbf{h}_n\|_V \le \frac{1}{C\delta_0}, \quad \|\mathbf{h}_n\|_{Q_n'} \to 0. \tag{3.71}$$

Since $(\mathbf{h}_n)$ is uniformly bounded in V, and by the compactness of the embedding of V into H, we conclude that there exists a subsequence $(\mathbf{h}_n)$ that converges to h strongly in H. We now show that $\mathbf{h}_n \to 0$ in H which will be contradiction with the assumption $\|\mathbf{h}_n\|_H = 1$.

Conclusion. We have checked that all the assumptions from Theorem 3.14 hold, so we can finally conclude that $\{(\mathbf{u}_N, \mathbf{v}_N, \mathbf{k}_N, \mathbf{z}_N)\}$ is relatively compact in $L^2(0,T;H)$. $\square$

We summarize the strong convergence results obtained in Subsection 3.8.1 and Theorem 3.15. We have shown that there exist subsequences $(\mathbf{u}_N)_{N\in\mathbb{N}}, (\boldsymbol{\eta}_N)_{N\in\mathbb{N}}, (\tilde{\boldsymbol{\eta}}_N)_{N\in\mathbb{N}}, (\mathbf{v}_N)_{N\in\mathbb{N}}, (\mathbf{k}_N)_{N\in\mathbb{N}}, (\mathbf{z}_N)_{N\in\mathbb{N}}$ such that

$$\begin{aligned}
\mathbf{u}_N &\to \mathbf{u} \text{ in } L^2(0,T;L^2(\Omega_M)),\\
\tau_{\Delta t}\hat{\mathbf{u}}_N &\to \mathbf{u} \text{ in } L^2(0,T;L^2(\Omega_M)),\\
\boldsymbol{\eta}_N &\to \boldsymbol{\eta} \text{ in } L^\infty(0,T;C(\overline{\omega})),\\
\tilde{\boldsymbol{\eta}}_N &\to \tilde{\boldsymbol{\eta}} \text{ in } L^\infty(0,T;C(\overline{\omega})),\\
\mathbf{v}_N &\to \mathbf{v} \text{ in } L^2(0,T;L^2(\omega))\\
\tau_{\Delta t}\mathbf{v}_N &\to \mathbf{v} \text{ in } L^2(0,T;L^2(\omega)),\\
\mathbf{k}_N &\to \mathbf{k} \text{ in } L^2(0,T;H^{-s}(\mathcal{N})),\\
\mathbf{z}_N &\to \mathbf{z} \text{ in } L^2(0,T;H^{-s}(\mathcal{N})).
\end{aligned} \tag{3.72}$$

The statements about convergence of $(\tau_{\Delta t}\hat{\mathbf{u}}_N)_{N\in\mathbb{N}}$ and $(\tau_{\Delta t}\mathbf{v}_N)_{N\in\mathbb{N}}$ follow directly from Statement 3 of Theorem 3.6. We conclude this section by stating one last convergence result that will be used in the next section to prove that the limiting functions satisfy the weak formulation (3.25) of the full FSI problem, i.e.

$$\tau_{\Delta t}\mathbf{u}_N \to \mathbf{u} \text{ in } L^2(0,T;L^2(\Omega_M)).$$

3.9 The limiting problem and main result

Next we want to show that the limiting functions satisfy the weak formulation (3.25) of the full fluid-mesh-shell interaction problem. We need to consider what happens in the limit as $N \to \infty$, that is, as $\Delta t \to 0$.

3.9.1 Construction of the test functions

We begin by recalling that the test functions $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ for the limiting problem are defined by the test space $\mathcal{Q}(0,T)$, which depends on $\boldsymbol{\eta}$. Similarly, the test spaces for the approximate problems depend on N through the dependence on $\boldsymbol{\eta}_N$. The fact that the velocity test functions depend on N presents a technical difficulty when passing to the limit as $N \to \infty$. For that reason, our goal is to construct test functions, both for the limiting problem and for the approximate problems, which are smooth, independent of N and divergence free.

We start by taking the test function $\boldsymbol{\psi} \in C_c^1([0,T);H^2(\omega))$, and define its extension on

Ω_M to be the function $\tilde{\boldsymbol{v}}$ such that $\tilde{\boldsymbol{v}}|_\Gamma \circ \boldsymbol{\varphi} = \boldsymbol{\psi}$. Then $\tilde{\boldsymbol{v}} \in C_c^1([0,T); H^2(\Omega_M))$. Using the function $\tilde{\boldsymbol{v}}$ and mappings $\boldsymbol{\phi}^n$, we define the test functions associated to the corresponding "discrete" physical domains:

$$\boldsymbol{v}_N^n = \begin{cases} \tilde{\boldsymbol{v}} \circ (\boldsymbol{\phi}^n)^{-1}, \text{ in } \Omega^n, \\ \tilde{\boldsymbol{v}} \circ (\tilde{\boldsymbol{\phi}}^n)^{-1}, \text{ in } \Omega_M \backslash \Omega^n, \end{cases}$$

where mapping $\tilde{\boldsymbol{\phi}}^n$ is the extension of the mapping $\boldsymbol{\phi}^n$ to the maximal domain Ω_M, as introduced in Lemma 3.16. It is easy to check that $\boldsymbol{v}_N^n \in H^1(\Omega_M), \forall n = 1, \dots, N$. Furthermore, we also define the test function associated to the "continuous" physical domain:

$$\boldsymbol{v} = \begin{cases} \tilde{\boldsymbol{v}} \circ (\boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}})^{-1}, \text{ in } \Omega^{\tilde{\boldsymbol{\eta}}}(t), \\ \tilde{\boldsymbol{v}} \circ (\tilde{\boldsymbol{\phi}}^{\tilde{\boldsymbol{\eta}}})^{-1}, \text{ in } \Omega_M \backslash \Omega^{\tilde{\boldsymbol{\eta}}}(t), \end{cases}$$

where mapping $\tilde{\boldsymbol{\phi}}^{\tilde{\boldsymbol{\eta}}}$ is the extension of the mapping $\boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}}$ to the maximal domain Ω_M. It is clear that $\boldsymbol{v} \in H^1(\Omega_M)$. We emphasize that the test functions $\boldsymbol{v}_N^n$ and $\boldsymbol{v}$ depend on the choice of the test function $\boldsymbol{\psi}$. However, for the simplicity of notation, we will not explicitly write that dependence.

From the uniform convergence of $\tilde{\boldsymbol{\eta}}_N$, we obtain that

$$\boldsymbol{v}_N \to \boldsymbol{v} \text{ uniformly in } (0,T) \times \Omega_M, \tag{3.73}$$

where $\boldsymbol{v}_N = (\boldsymbol{v}_N^1, \boldsymbol{v}_N^2, \dots, \boldsymbol{v}_N^N)$. Using the chain rule, and the fact that $\nabla \boldsymbol{\eta}_N \to \nabla \boldsymbol{\eta}$ in $L^2(0,T; L^p(\omega))$, one can see that

$$\nabla \boldsymbol{v}_N \to \nabla \boldsymbol{v} \text{ in } L^2(0,T; L^p(\Omega_M)), \quad p < \infty.$$

Even though the functions $\tilde{\boldsymbol{v}}$ are smooth both in the spatial and time variable, the functions $\boldsymbol{v}_N$ are discontinuous at $n\Delta t$ because $\boldsymbol{\phi}^n(z,r,\theta) = \boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}}(n\Delta t, z, r, \theta) = (z, (R + \tilde{\eta}_N^n)r, \theta)$ is a step function in time. For that reason, we define an approximate test function $\overline{\boldsymbol{v}}_N$ to be continuous, linear on each subinterval $[(n-1)\Delta t, n\Delta t], n = 1, \dots, N$, and such that

$$\overline{\boldsymbol{v}}_N(n\Delta t, \cdot) = \boldsymbol{v}_N(n\Delta t, \cdot).$$

Using the strong convergence of the shell velocity in $L^2(0,T; L^2(\omega))$, we get that

$$\partial_t \overline{\boldsymbol{v}}_N \to \partial_t \boldsymbol{v} \text{ in } L^2(0,T; L^p(\Omega_M)) \quad p < 2.$$

Unfortunately, neither $\boldsymbol{v}_N$ nor $\boldsymbol{v}$ are divergence free. In order to get around this difficulty, we will again use Theorem III.3.1. from [43], just like in Property C2. First, we define mappings $\mathbf{a}_N^n : \Omega^n \to \mathbb{R}^3$ satisfying the following conditions:

(i) $\operatorname{supp} \mathbf{a}_N^n \subseteq \Omega^n$,

(ii) $\displaystyle\int_{\Omega^n} \nabla \cdot (\boldsymbol{v}_N^n + \mathbf{a}_N^n) = 0$,

(iii) $\|\mathbf{a}_N^n\|_{H^1(\Omega^n)} \leq C\|\boldsymbol{v}_N^n\|_{H^1(\Omega_M)}$,

and mappings $\mathbf{b}_N^n : (\Omega_M \backslash \Omega^n) \to \mathbb{R}^3$ satisfying the following conditions:

(i) $\operatorname{supp} \mathbf{b}_N^n \subseteq \Omega_M \backslash \Omega^n$,

(ii) $\displaystyle\int_{\Omega_M \backslash \Omega^n} \nabla \cdot (\boldsymbol{v}_N^n + \mathbf{b}_N^n) = 0$,

(iii) $\|\mathbf{b}_N^n\|_{H^1(\Omega_M \backslash \Omega^n)} \leq C\|\boldsymbol{v}_N^n\|_{H^1(\Omega_M)}$,

in the same way as in Lemma 3.16. Furthermore, the same reasoning as in Lemma 3.16 tells us that we can decompose both Ω^n and $\Omega_M \backslash \Omega^n$ as a union of exactly 4 star-shaped domains (with respect to fixed balls). We are now in position to apply Theorem III.3.1. from [43] to see that there exists a function $\mathbf{A}_N^n$ such that:

$$\nabla \cdot \mathbf{A}_N^n = -\nabla \cdot (\boldsymbol{v}_N^n + \mathbf{a}_N^n)$$

with

$$\|\mathbf{A}_N^n\|_{H^1(\Omega_M)} \leq C\|\boldsymbol{v}_N^n + \mathbf{a}_N^n\|_{H^1(\Omega_M)},$$

and a function $\mathbf{B}_N^n$ such that:

$$\nabla \cdot \mathbf{B}_N^n = -\nabla \cdot (\boldsymbol{v}_N^n + \mathbf{b}_N^n)$$

with

$$\|\mathbf{B}_N^n\|_{H^1(\Omega_M)} \leq C\|\boldsymbol{v}_N^n + \mathbf{b}_N^n\|_{H^1(\Omega_M)}.$$

Finally, if we set

$$\boldsymbol{v}_N(\boldsymbol{\psi}) = \begin{cases} \boldsymbol{v}_N^n + \mathbf{a}_N^n + \mathbf{A}_N^n, & \text{in } \Omega^n, \\ \boldsymbol{v}_N^n + \mathbf{b}_N^n + \mathbf{B}_N^n, & \text{in } \Omega_M \backslash \Omega^n, \end{cases}$$

we have that $\boldsymbol{v}_N(\boldsymbol{\psi})$ is a smooth, divergence free function on the maximal domain Ω_M. In the same way, we can construct a divergence free extension $\boldsymbol{v}(\boldsymbol{\psi})$ of the test function $\boldsymbol{v}$ corresponding to the limiting problem:

$$\boldsymbol{v}(\boldsymbol{\psi}) = \begin{cases} \boldsymbol{v} + \mathbf{a} + \mathbf{A}, & \text{in } \Omega^n, \\ \boldsymbol{v} + \mathbf{b} + \mathbf{B}, & \text{in } \Omega_M \backslash \Omega^n, \end{cases}$$

with $\|\mathbf{a}\|_{H^1(\Omega^n)}, \|\mathbf{A}\|_{H^1(\Omega^n)} \leq C\|\boldsymbol{v}\|_{H^1(\Omega_M)}$ and $\|\mathbf{b}\|_{H^1(\Omega_M \backslash \Omega^n)}, \|\mathbf{B}\|_{H^1(\Omega_M \backslash \Omega^n)} \leq C\|\boldsymbol{v}\|_{H^1(\Omega_M)}$. Due to the fact that $\mathbf{A}_N^n, \mathbf{A}, \mathbf{B}_N^n, \mathbf{B}$ are solutions of equation (3.69), with corresponding right-hand sides $-\nabla \cdot (\boldsymbol{v}_N^n + \mathbf{a}_N^n), -\nabla \cdot (\boldsymbol{v} + \mathbf{a}), -\nabla \cdot (\boldsymbol{v}_N^n + \mathbf{b}_N^n), -\nabla \cdot (\boldsymbol{v} + \mathbf{b})$, which belong to $L^p(\Omega_M)$, we can write their explicit formulas by using Bogowskii construction, and use Theorem III.3.3 from [43] to obtain additional regularity on $\mathbf{A}_N^n, \mathbf{A}, \mathbf{B}_N^n, \mathbf{B}$:

$$\begin{aligned}\|\boldsymbol{v}_N(\boldsymbol{\psi}) - \boldsymbol{v}(\boldsymbol{\psi})\|_{W^{1,p}(\Omega_M)} &= \|\boldsymbol{v}_N^n + \mathbf{a}_N^n + \mathbf{A}_N^n - \boldsymbol{v} - \mathbf{a} - \mathbf{A}\|_{W^{1,p}(\Omega^n)} \\ &+ \|\boldsymbol{v}_N^n + \mathbf{b}_N^n + \mathbf{B}_N^n - \boldsymbol{v} - \mathbf{b} - \mathbf{B}\|_{W^{1,p}(\Omega_M \setminus \Omega^n)} \\ &\leq \|(\boldsymbol{v}_N^n + \mathbf{a}_N^n) - (\boldsymbol{v} + \mathbf{a})\|_{W^{1,p}(\Omega^n)} + \|\mathbf{A}_N^n - \mathbf{A}\|_{W^{1,p}(\Omega^n)} \\ &+ \|(\boldsymbol{v}_N^n + \mathbf{b}_N^n) - (\boldsymbol{v} + \mathbf{b})\|_{W^{1,p}(\Omega_M \setminus \Omega^n)} + \|\mathbf{B}_N^n - \mathbf{B}\|_{W^{1,p}(\Omega_M \setminus \Omega^n)}\end{aligned}$$

The uniform convergence of $\boldsymbol{v}_N \to \boldsymbol{v}$, provides that the right-hand side of the previous inequality tends to 0. Furthermore, using the Sobolev embedding of $W^{1,p}(\Omega_M)$ to $C(\overline{\Omega}_M)$, for $p > 3$, we obtain that

$$\boldsymbol{v}_N(\boldsymbol{\psi}) \to \boldsymbol{v}(\boldsymbol{\psi}) \text{ uniformly in } (0,T) \times \Omega_M.$$

Additionally, by using Remark III.3.3 from [43], we can show that

$$\begin{aligned}&\nabla \boldsymbol{v}_N(\boldsymbol{\psi}) \to \nabla \boldsymbol{v}(\boldsymbol{\psi}) \text{ in } L^2(0,T; L^p(\Omega_M)), \quad p < \infty, \\ &\partial_t \overline{\boldsymbol{v}}_N(\boldsymbol{\psi}) \to \partial_t \boldsymbol{v}(\boldsymbol{\psi}) \text{ in } L^2(0,T; L^p(\Omega_M)), \quad p < 2.\end{aligned}$$

To conclude, for any test function $(\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in \mathcal{Q}(0,T)$, the fluid velocity component $\boldsymbol{v}$ can be written as $(\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{\psi}) + \boldsymbol{v}(\boldsymbol{\psi}), \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$, where $\boldsymbol{v} - \boldsymbol{v}(\boldsymbol{\psi})$ can be approximated by divergence free functions $\boldsymbol{v}_0$, which have compact support in $\Omega^{\tilde{\boldsymbol{\eta}}}(t) \cup \Gamma_{in} \cup \Gamma_{out}$. Therefore, one can easily see that the functions $(\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) = (\boldsymbol{v}_0 + \boldsymbol{v}(\boldsymbol{\psi}), \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ are dense in $\mathcal{Q}(0,T)$, with $\nabla \cdot \boldsymbol{v} = 0, \forall \boldsymbol{v}$. The corresponding test functions for approximate problem have the same form, i.e. $(\boldsymbol{v}_N, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) = (\boldsymbol{v}_0 + \boldsymbol{v}_N(\boldsymbol{\psi}), \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$.

3.9.2 Passing to the limit

We start by writing the weak formulation of the coupled, semi-discretized problem. For this purpose, take $(\boldsymbol{\psi}(t), \boldsymbol{\xi}(t), \boldsymbol{\zeta}(t))$ as the test function in the structure subproblem (3.34), and integrate with respect to t from $n\Delta t$ to $(n+1)\Delta t$. Then, take the $(\boldsymbol{v}_N(t), \boldsymbol{\psi}(t))$, where $\boldsymbol{v}_N(t) = \boldsymbol{v}_0 + \boldsymbol{v}_N(\boldsymbol{\psi}) \in \mathcal{H}_F$, as the test functions in the fluid subproblem (3.40), and again integrate over the same time interval. Add the two equations together to obtain:

$$\begin{aligned}&\rho_F \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} \frac{\mathbf{u}_N^{n+1} - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \boldsymbol{v}_N^{n+1} + \frac{\rho_F}{2} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}_N^n \cdot \boldsymbol{v}_N^{n+1}) \\ &+ \frac{\rho_F}{2} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} \Big[((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla) \mathbf{u}_N^{n+1} \cdot \boldsymbol{v}_N^{n+1} - ((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \Big] \\ &+ 2\mu_F \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}_N^{n+1}) : \mathbf{D}(\boldsymbol{v}_N^{n+1}) + \rho_K h \int_{n\Delta t}^{(n+1)\Delta t} \int_\omega \frac{\mathbf{v}_N^{n+1} - \mathbf{v}_N^n}{\Delta t} \cdot \boldsymbol{\psi} R\end{aligned}$$

$$
\begin{aligned}
&+ \int_{n\Delta t}^{(n+1)\Delta t} a_K(\boldsymbol{\eta}_N^{n+1}, \boldsymbol{\psi}) + \rho_S \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{(\mathbf{k}_N^{n+1})_i - (\mathbf{k}_N^n)_i}{\Delta t} \cdot \boldsymbol{\xi}_i \\
&+ \rho_S \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{(\mathbf{z}_N^{n+1})_i - (\mathbf{z}_N^n)_i}{\Delta t} \cdot \boldsymbol{\zeta}_i + \int_{n\Delta t}^{(n+1)\Delta t} a_S(\mathbf{w}_N^{n+1}, \boldsymbol{\zeta}) \\
&= \int_{n\Delta t}^{(n+1)\Delta t} P_{in}^n \int_{\Gamma_{in}} \upsilon_z - \int_{n\Delta t}^{(n+1)\Delta t} P_{out}^n \int_{\Gamma_{out}} \upsilon_z.
\end{aligned}
$$

We take the sum from $n = 0, \ldots, N-1$ to obtain:

$$
\begin{aligned}
&\rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} \frac{\mathbf{u}_N^{n+1} - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \boldsymbol{\upsilon}_N^{n+1} + \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}_N^n \cdot \boldsymbol{\upsilon}_N^{n+1}) \\
&+ \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} \Big[((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla) \mathbf{u}_N^{n+1} \cdot \boldsymbol{\upsilon}_N^{n+1} - ((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla) \boldsymbol{\upsilon}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \Big] \\
&+ 2\mu_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega^{n+1}} \mathbf{D}(\mathbf{u}_N^{n+1}) : \mathbf{D}(\boldsymbol{\upsilon}_N^{n+1}) + \rho_K h \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\omega \frac{\overline{\mathbf{v}}_N - \tau_{\Delta t} \overline{\mathbf{v}}_N}{\Delta t} \cdot \boldsymbol{\psi} R \\
&+ \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} a_K(\boldsymbol{\eta}_N(t), \boldsymbol{\psi}) + \rho_S \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{n_E} A_i \int_0^{l_i} \frac{(\overline{\mathbf{k}}_N)_i - \tau_{\Delta t}(\overline{\mathbf{k}}_N)_i}{\Delta t} \cdot \boldsymbol{\xi}_i \\
&+ \rho_S \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \sum_{i=1}^{n_E} \int_0^{l_i} M_i \frac{(\overline{\mathbf{z}}_N)_i - \tau_{\Delta t}(\overline{\mathbf{z}}_N)_i}{\Delta t} \cdot \boldsymbol{\zeta}_i + \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} a_S(\mathbf{w}_N(t), \boldsymbol{\zeta}) \\
&= \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} P_{in}^N \int_{\Gamma_{in}} \upsilon_z - \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} P_{out}^N \int_{\Gamma_{out}} \upsilon_z,
\end{aligned}
$$

where we have used the definition of $\boldsymbol{\eta}_N$ and $\mathbf{w}_N$ as a piecewise constant approximations (defined in (3.43)), and the definition of $\overline{\mathbf{v}}_N, \overline{\mathbf{k}}_N$ an $\overline{\mathbf{z}}_N$ as a piecewise linear approximations (defined in (3.45)). The terms that include the shell and the mesh unknowns can be written as

$$
\begin{aligned}
&\rho_K h \int_0^T \int_\omega \partial_t \overline{\mathbf{v}}_N \cdot \boldsymbol{\psi} R + \int_0^T a_K(\boldsymbol{\eta}_N, \boldsymbol{\psi}) + \rho_S \int_0^T \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t (\overline{\mathbf{k}}_N)_i \cdot \boldsymbol{\xi}_i \\
&+ \rho_S \int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t (\overline{\mathbf{z}}_N)_i \cdot \boldsymbol{\zeta}_i + \int_0^T a_S(\mathbf{w}_N, \boldsymbol{\zeta}),
\end{aligned}
$$

and integration by parts with respect to time gives:

$$
\begin{aligned}
&- \rho_K h \int_0^T \int_\omega \overline{\mathbf{v}}_N \cdot \partial_t \boldsymbol{\psi} R - \rho_K h \int_\omega \mathbf{v}_0 \cdot \boldsymbol{\psi}(0) R + \int_0^T a_K(\boldsymbol{\eta}_N, \boldsymbol{\psi}) \\
&- \rho_S \int_0^T \sum_{i=1}^{n_E} A_i \int_0^{l_i} (\overline{\mathbf{k}}_N)_i \cdot \partial_t \boldsymbol{\xi}_i - \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{k}_{0i} \cdot \boldsymbol{\xi}_i(0) \\
&- \rho_S \int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} M_i (\overline{\mathbf{z}}_N)_i \cdot \partial_t \boldsymbol{\zeta}_i - \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{z}_{0i} \cdot \boldsymbol{\zeta}_i(0) + \int_0^T a_S(\mathbf{w}_N, \boldsymbol{\zeta}).
\end{aligned}
$$

We now leave the structure terms aside and take care of the fluid part. First of all, the characteristic functions, as defined in (3.48), enable us to rewrite the fluid part on the maximal domain Ω_M. In order to do that, we set $\chi_N(t,\cdot) = \chi_N^{n+1}$, for $t \in (n\Delta t, (n+1)\Delta t]$, and write:

$$\rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \frac{\mathbf{u}_N^{n+1} - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \boldsymbol{v}_N^{n+1} + \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}_N^n \cdot \boldsymbol{v}_N^{n+1})$$
$$+ \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \Big[((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla) \mathbf{u}_N^{n+1} \cdot \boldsymbol{v}_N^{n+1} - ((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \Big]$$
$$+ 2\mu_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \mathbf{D}(\mathbf{u}_N^{n+1}) : \mathbf{D}(\boldsymbol{v}_N^{n+1}).$$

Let us take care of each term separately. The first term does not have the right form, so we add and subtract $\mathbf{u}_N^n$ from the numerator to obtain:

$$\rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{\mathbf{u}_N^{n+1} - \mathbf{u}_N^n}{\Delta t} \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1} + \rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{\mathbf{u}_N^n - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1}. \tag{3.74}$$

We now use the summation by parts formula (discrete analogue of the integration by parts formula) to take care of the first term in (3.74):

$$\rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{\mathbf{u}_N^{n+1} - \mathbf{u}_N^n}{\Delta t} \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1} = \rho_F \int_{\Omega_M} \mathbf{u}_N^N \cdot \chi_N^N \boldsymbol{v}_N^N - \rho_F \int_{\Omega_M} \mathbf{u}_N^0 \cdot \chi_N^1 \boldsymbol{v}_N^1$$
$$- \rho_F \sum_{n=1}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{1}{\Delta t} \mathbf{u}_N^n \cdot (\chi_N^{n+1} \boldsymbol{v}_N^{n+1} - \chi_N^n \boldsymbol{v}_N^n)$$
$$= -\rho_F \int_{\Omega_M} \mathbf{u}_N^0 \cdot \chi_N^1 \boldsymbol{v}_N^1 - \rho_F \sum_{n=1}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{1}{\Delta t} \mathbf{u}_N^n \cdot \chi_N^{n+1} (\boldsymbol{v}_N^{n+1} - \boldsymbol{v}_N^n)$$
$$- \rho_F \sum_{n=1}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{1}{\Delta t} \mathbf{u}_N^n \cdot (\chi_N^{n+1} - \chi_N^n) \boldsymbol{v}_N^n$$
$$= -\rho_F \int_{\Omega_M} \chi_N^1 \mathbf{u}_N^0 \cdot \boldsymbol{v}_N^1 - \rho_F \int_0^T \int_{\Omega_M} \chi_N \tau_{\Delta t} \mathbf{u}_N \cdot \partial_t \boldsymbol{v}_N$$
$$- \rho_F \sum_{n=1}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\omega} \frac{R}{\Delta t} \int_{R+\tilde{\eta}^n}^{R+\tilde{\eta}^{n+1}} \mathbf{u}_N^n \cdot \boldsymbol{v}_N^n$$
$$= -\rho_F \int_{\Omega_M} \chi_N^1 \mathbf{u}_N^0 \cdot \boldsymbol{v}_N^1 - \rho_F \int_0^T \int_{\Omega_M} \chi_N \tau_{\Delta t} \mathbf{u}_N \cdot \partial_t \boldsymbol{v}_N - \rho_F \int_0^T \int_{\omega} \partial_t \tilde{\eta}_N R (\tau_{\Delta t} \mathbf{u}_N \cdot \tau_{\Delta t} \boldsymbol{v}_N).$$

Notice that in the last equality we used the mean value theorem for integrals. To deal with the second term in (3.74), we recall that $\hat{\mathbf{u}}_N^n$ was defined in (3.39) as a composition of $\mathbf{u}_N^n$ with $\mathbf{A}^{n+1,n}$, and calculate:

$$\rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{\mathbf{u}_N^n - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1}$$
$$= \rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{1}{\Delta t} \left(\mathbf{u}_N^n(z,r,\theta) - \mathbf{u}_N^n(z, \frac{R+\tilde{\eta}_N^n}{R+\tilde{\eta}_N^{n+1}} r, \theta) \right) \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1}$$

$$
\begin{aligned}
&= \rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{1}{\Delta t} \left((\nabla \mathbf{u}_N^n) \frac{\tilde{\eta}_N^{n+1} - \tilde{\eta}_N^n}{R + \tilde{\eta}_N^{n+1}} r \mathbf{e}_r \right) \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1} \\
&= \rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} (\nabla \mathbf{u}_N^n) \mathbf{s}^{n+1,n} \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1} \\
&= \rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} (\mathbf{s}^{n+1,n} \cdot \nabla) \mathbf{u}_N^n \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1} \\
&= \rho_F \int_0^T \int_{\Omega_M} (\mathbf{s}_N \cdot \nabla) \tau_{\Delta t} \mathbf{u}_N \cdot \chi_N \boldsymbol{v}_N \\
&= \rho_F \int_0^T \int_{\Omega_M} \chi_N (\mathbf{s}_N \cdot \nabla) \tau_{\Delta t} \mathbf{u}_N \cdot \boldsymbol{v}_N,
\end{aligned}
$$

where

$$
\mathbf{s}_N = \frac{\tilde{\eta}_N - \tau_{\Delta t} \tilde{\eta}_N}{\Delta t (R + \tilde{\eta}_N)} r \mathbf{e}_r = \frac{\partial_t \tilde{\eta}_N}{R + \tilde{\eta}_N} r \mathbf{e}_r.
$$

We rewrite the convective part in the following way:

$$
\begin{aligned}
&\frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \left[(\hat{\mathbf{u}}_N^n \cdot \nabla) \mathbf{u}_N^{n+1} \cdot \boldsymbol{v}_N^{n+1} - (\hat{\mathbf{u}}_N^n \cdot \nabla) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \right] \\
&+ \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \left[(\mathbf{s}^{n+1,n} \cdot \nabla) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} - (\mathbf{s}^{n+1,n} \cdot \nabla) \mathbf{u}_N^{n+1} \cdot \boldsymbol{v}_N^{n+1} \right].
\end{aligned}
\tag{3.75}
$$

Furthermore, we calculate:

$$
\begin{aligned}
&\int_{\Omega_M} \chi_N^{n+1} (\mathbf{s}^{n+1,n} \cdot \nabla) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} = \int_{\Gamma_M} \chi_N^{n+1} (\mathbf{s}^{n+1,n} \cdot \mathbf{n}) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \\
&- \int_{\Omega_M} \chi_N^{n+1} (\nabla \cdot \mathbf{s}^{n+1,n}) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} - \int_{\Omega_M} \chi_N^{n+1} (\mathbf{s}^{n+1,n} \cdot \nabla) \mathbf{u}_N^{n+1} \cdot \boldsymbol{v}_N^{n+1}.
\end{aligned}
$$

By using the definition of $\mathbf{s}^{n+1,n}$, the boundary term can be rewritten as follows:

$$
\begin{aligned}
\int_{\Gamma_M} \chi_N^{n+1} (\mathbf{s}^{n+1,n} \cdot \mathbf{n}) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} &= \int_{\Gamma^{n+1}} (\mathbf{s}^{n+1,n} \cdot \mathbf{n}) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \\
&= \int_{\Gamma^{n+1}} \left(\frac{\tilde{\eta}_N^{n+1} - \tilde{\eta}_N^n}{\Delta t (R + \tilde{\eta}_N^{n+1})} r \mathbf{e}_r \cdot \mathbf{n} \right) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \\
&= \int_{\Gamma} \left(\frac{\tilde{\eta}_N^{n+1} - \tilde{\eta}_N^n}{\Delta t} \right) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \\
&= \int_{\omega} \left(\frac{\tilde{\eta}_N^{n+1} - \tilde{\eta}_N^n}{\Delta t} R \right) \boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1}.
\end{aligned}
$$

By inserting the previous calculations into (3.75), we obtain that the convective term is equal to:

$$
\begin{aligned}
&= \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N \left[(\tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \nabla) \mathbf{u}_N \cdot \boldsymbol{v}_N - (\tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \nabla) \boldsymbol{v}_N \cdot \mathbf{u}_N \right] \\
&\quad + \frac{\rho_F}{2} \int_0^T \int_{\omega} \partial_t \tilde{\eta}_N R \boldsymbol{v}_N \cdot \mathbf{u}_N - \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N (\nabla \cdot \mathbf{s}_N) \boldsymbol{v}_N \cdot \mathbf{u}_N
\end{aligned}
$$

$$- \rho_F \int_0^T \int_{\Omega_M} \chi_N (\mathbf{s}_N \cdot \nabla)\mathbf{u}_N \cdot \boldsymbol{v}_N.$$

We are finally ready to see what we have obtained from the fluid part:

$$\begin{aligned}
&\rho_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \frac{\mathbf{u}_N^{n+1} - \hat{\mathbf{u}}_N^n}{\Delta t} \cdot \chi_N^{n+1} \boldsymbol{v}_N^{n+1} + \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} (\nabla \cdot \mathbf{s}^{n+1,n})(\hat{\mathbf{u}}_N^n \cdot \boldsymbol{v}_N^{n+1}) \\
&+ \frac{\rho_F}{2} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \Big[((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\mathbf{u}_N^{n+1} \cdot \boldsymbol{v}_N^{n+1} - ((\hat{\mathbf{u}}_N^n - \mathbf{s}^{n+1,n}) \cdot \nabla)\boldsymbol{v}_N^{n+1} \cdot \mathbf{u}_N^{n+1} \Big] \\
&+ 2\mu_F \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_M} \chi_N^{n+1} \mathbf{D}(\mathbf{u}_N^{n+1}) : \mathbf{D}(\boldsymbol{v}_N^{n+1}) \\
&= -\rho_F \int_{\Omega_M} \chi_N^1 \mathbf{u}_N^0 \cdot \boldsymbol{v}_N^1 - \rho_F \int_0^T \int_{\Omega_M} \chi_N \tau_{\Delta t} \mathbf{u}_N \cdot \partial_t \boldsymbol{v}_N - \rho_F \int_0^T \int_\omega \partial_t \tilde{\eta}_N R (\tau_{\Delta t} \mathbf{u}_N \cdot \tau_{\Delta t} \boldsymbol{v}_N) \\
&+ \rho_F \int_0^T \int_{\Omega_M} \chi_N (\mathbf{s}_N \cdot \nabla) \tau_{\Delta t} \mathbf{u}_N \cdot \boldsymbol{v}_N + \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N (\nabla \cdot \mathbf{s}_N) \tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \boldsymbol{v}_N \\
&+ \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N \left[(\tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \nabla) \mathbf{u}_N \cdot \boldsymbol{v}_N - (\tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \nabla) \boldsymbol{v}_N \cdot \mathbf{u}_N \right] \\
&+ \frac{\rho_F}{2} \int_0^T \int_\omega \partial_t \tilde{\eta}_N R \boldsymbol{v}_N \cdot \mathbf{u}_N - \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N (\nabla \cdot \mathbf{s}_N) \boldsymbol{v}_N \cdot \mathbf{u}_N \\
&- \rho_F \int_0^T \int_{\Omega_M} \chi_N (\mathbf{s}_N \cdot \nabla) \mathbf{u}_N \cdot \boldsymbol{v}_N + 2\mu_F \int_0^T \int_{\Omega_M} \chi_N \mathbf{D}(\mathbf{u}_N) : \mathbf{D}(\boldsymbol{v}_N).
\end{aligned}$$

Now the weak formulation of the full, coupled problem can be rewritten as follows:

$$\begin{aligned}
&- \rho_F \int_{\Omega_M} \mathbf{u}_N^0 \chi_N^1 \cdot \boldsymbol{v}_N^1 - \rho_F \int_0^T \int_{\Omega_M} \chi_N \tau_{\Delta t} \mathbf{u}_N \cdot \partial_t \boldsymbol{v}_N - \rho_F \int_0^T \int_\omega \partial_t \tilde{\eta}_N R (\tau_{\Delta t} \mathbf{u}_N \cdot \tau_{\Delta t} \boldsymbol{v}_N) \\
&+ \rho_F \int_0^T \int_{\Omega_M} \chi_N (\mathbf{s}_N \cdot \nabla) \tau_{\Delta t} \mathbf{u}_N \cdot \boldsymbol{v}_N + \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N (\nabla \cdot \mathbf{s}_N) \tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \boldsymbol{v}_N \\
&+ \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N \left[(\tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \nabla) \mathbf{u}_N \cdot \boldsymbol{v}_N - (\tau_{\Delta t} \hat{\mathbf{u}}_N \cdot \nabla) \boldsymbol{v}_N \cdot \mathbf{u}_N \right] \\
&+ \frac{\rho_F}{2} \int_0^T \int_\omega \partial_t \tilde{\eta}_N R \boldsymbol{v}_N \cdot \mathbf{u}_N - \frac{\rho_F}{2} \int_0^T \int_{\Omega_M} \chi_N (\nabla \cdot \mathbf{s}_N) \boldsymbol{v}_N \cdot \mathbf{u}_N \\
&- \rho_F \int_0^T \int_{\Omega_M} \chi_N (\mathbf{s}_N \cdot \nabla) \mathbf{u}_N \cdot \boldsymbol{v}_N + 2\mu_F \int_0^T \int_{\Omega_M} \chi_N \mathbf{D}(\mathbf{u}_N) : \mathbf{D}(\boldsymbol{v}_N) \\
&- \rho_K h \int_0^T \int_\omega \overline{\mathbf{v}}_N \cdot \partial_t \boldsymbol{\psi} R - \rho_K h \int_\omega \mathbf{v}_0 \cdot \boldsymbol{\psi}(0) R + \int_0^T a_K(\boldsymbol{\eta}_N, \boldsymbol{\psi}) \\
&- \rho_S \int_0^T \sum_{i=1}^{n_E} A_i \int_0^{l_i} (\overline{\mathbf{k}}_N)_i \cdot \partial_t \boldsymbol{\xi}_i - \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{k}_{0i} \cdot \boldsymbol{\xi}_i(0) \\
&- \rho_S \int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} M_i (\overline{\mathbf{z}}_N)_i \cdot \partial_t \boldsymbol{\zeta}_i - \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{z}_{0i} \cdot \boldsymbol{\zeta}_i(0) + \int_0^T a_S(\mathbf{w}_N, \boldsymbol{\zeta}) \\
&= \int_0^T P_{in}^N(t) \int_{\Gamma_{in}} \upsilon_z - \int_0^T P_{out}^N(t) \int_{\Gamma_{out}} \upsilon_z.
\end{aligned}$$

Using the strong convergence results summarized in (3.72), we can pass to the limit in all the terms:

$$
\begin{aligned}
&-\rho_F\int_\Omega \mathbf{u}_0\cdot\boldsymbol{v}(0)-\rho_F\int_0^T\int_{\Omega_M}\chi\mathbf{u}\cdot\partial_t\boldsymbol{v}-\rho_F\int_0^T\int_\omega \partial_t\tilde{\eta}R(\mathbf{u}\cdot\boldsymbol{v})\\
&+\rho_F\int_0^T\int_{\Omega_M}\chi\Big(\frac{\partial_t\tilde{\eta}}{R+\tilde{\eta}}r\mathbf{e}_r\cdot\nabla\Big)\mathbf{u}\cdot\boldsymbol{v}+\frac{\rho_F}{2}\int_0^T\int_{\Omega_M}\chi\Big(\nabla\cdot\frac{\partial_t\tilde{\eta}}{R+\tilde{\eta}}r\mathbf{e}_r\Big)\mathbf{u}\cdot\boldsymbol{v}\\
&+\frac{\rho_F}{2}\int_0^T\int_{\Omega_M}\chi\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{v}-(\mathbf{u}\cdot\nabla)\boldsymbol{v}\cdot\mathbf{u}\right]\\
&+\frac{\rho_F}{2}\int_0^T\int_\omega \partial_t\tilde{\eta}R\boldsymbol{v}\cdot\mathbf{u}-\frac{\rho_F}{2}\int_0^T\int_{\Omega_M}\chi\Big(\nabla\cdot\frac{\partial_t\tilde{\eta}}{R+\tilde{\eta}}r\mathbf{e}_r\Big)\boldsymbol{v}\cdot\mathbf{u}\\
&-\rho_F\int_0^T\int_{\Omega_M}\chi\Big(\frac{\partial_t\tilde{\eta}}{R+\tilde{\eta}}r\mathbf{e}_r\cdot\nabla\Big)\mathbf{u}\cdot\boldsymbol{v}+2\mu_F\int_0^T\int_{\Omega_M}\chi\mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{v})\\
&-\rho_K h\int_0^T\int_\omega \mathbf{v}\cdot\partial_t\boldsymbol{\psi}R-\rho_K h\int_\omega \mathbf{v}_0\cdot\boldsymbol{\psi}(0)R+\int_0^T a_K(\boldsymbol{\eta},\boldsymbol{\psi})\\
&-\rho_S\int_0^T\sum_{i=1}^{n_E}A_i\int_0^{l_i}\mathbf{k}_i\cdot\partial_t\boldsymbol{\xi}_i-\rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\mathbf{k}_{0i}\cdot\boldsymbol{\xi}_i(0)\\
&-\rho_S\int_0^T\sum_{i=1}^{n_E}\int_0^{l_i}M_i(\overline{\mathbf{z}}_N)_i\cdot\partial_t\boldsymbol{\zeta}_i-\rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\mathbf{z}_{0i}\cdot\boldsymbol{\zeta}_i(0)+\int_0^T a_S(\mathbf{w},\boldsymbol{\zeta})\\
&=\int_0^T P_{in}(t)\int_{\Gamma_{in}}v_z-\int_0^T P_{out}(t)\int_{\Gamma_{out}}v_z.
\end{aligned}
$$

Using the definition of the characteristic function χ, we write the weak formulation on the physical domain $\Omega^{\boldsymbol{\eta}}(t)$:

$$
\begin{aligned}
&-\rho_F\int_0^T\int_{\Omega^{\boldsymbol{\eta}}(t)}\mathbf{u}\cdot\partial_t\boldsymbol{v}-\frac{\rho_F}{2}\int_0^T\int_\omega \partial_t\tilde{\eta}R(\mathbf{u}\cdot\boldsymbol{v})+\frac{\rho_F}{2}\int_0^T\int_{\Omega^{\boldsymbol{\eta}}(t)}b(t,\mathbf{u},\mathbf{u},\boldsymbol{v})\\
&+2\mu_F\int_0^T\int_{\Omega^{\boldsymbol{\eta}}(t)}\mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{v})-\rho_K h\int_0^T\int_\omega \mathbf{v}\cdot\partial_t\boldsymbol{\psi}R+\int_0^T a_K(\boldsymbol{\eta},\boldsymbol{\psi})\\
&-\rho_S\int_0^T\sum_{i=1}^{n_E}A_i\int_0^{l_i}\mathbf{k}_i\cdot\partial_t\boldsymbol{\xi}_i-\rho_S\int_0^T\sum_{i=1}^{n_E}\int_0^{l_i}M_i(\overline{\mathbf{z}}_N)_i\cdot\partial_t\boldsymbol{\zeta}_i+\int_0^T a_S(\mathbf{w},\boldsymbol{\zeta})\\
&=\int_0^T P_{in}(t)\int_{\Gamma_{in}}v_z-\int_0^T P_{out}(t)\int_{\Gamma_{out}}v_z+\rho_F\int_\Omega \mathbf{u}_0\cdot\boldsymbol{v}(0)\\
&+\rho_K h\int_\omega \mathbf{v}_0\cdot\boldsymbol{\psi}(0)R+\rho_S\sum_{i=1}^{n_E}A_i\int_0^{l_i}\mathbf{k}_{0i}\cdot\boldsymbol{\xi}_i(0)+\rho_S\sum_{i=1}^{n_E}\int_0^{l_i}M_i\mathbf{z}_{0i}\cdot\boldsymbol{\zeta}_i(0).
\end{aligned}
$$

To see that we obtained exactly the weak formulation (3.25), we have to rewrite the second term from the right-hand side, i.e. $\int_0^T\int_\omega \partial_t\tilde{\eta}R(\mathbf{u}\cdot\boldsymbol{v})$. Using the fact that $\tilde{\eta}(t,\tilde{z},\tilde{\theta})=\eta_r(t,z,\theta)$, it is not hard to check that the following equality holds true:

$$
\partial_t\tilde{\eta}=\partial_t\eta_r-\frac{\partial_z\eta_r}{1+\partial_z\eta_z}\cdot\partial_t\eta_z-\frac{\partial_\theta\eta_r}{1+\partial_\theta\eta_\theta}\cdot\partial_t\eta_\theta.
$$

Additionally, the outer normal $\mathbf{n}$ on $\Gamma^{\tilde{\boldsymbol{\eta}}}(t)$ is equal to $(-\partial_{\tilde{z}}\tilde{\eta}, 1, -\partial_{\tilde{\theta}}\tilde{\eta})$, and if we rewrite it in "original" coordinates, we obtain that the outer normal is equal to $\left(-\frac{\partial_z \eta_r}{1+\partial_z \eta_z}, 1, -\frac{\partial_\theta \eta_r}{1+\partial_\theta \eta_\theta}\right)$. This yields that on $\Gamma^{\boldsymbol{\eta}}(t)$ we have:

$$\partial_t \tilde{\eta} = \partial_t \boldsymbol{\eta} \cdot \mathbf{n}. \tag{3.76}$$

Finally, using (3.76) and the kinematic coupling condition on $\Gamma^{\boldsymbol{\eta}}(t)$, we obtain:

$$\begin{aligned}\int_0^T \int_\omega \partial_t \tilde{\eta} R(\mathbf{u}\cdot\boldsymbol{v}) &= \int_0^T \int_\Gamma \partial_t \tilde{\eta}(\mathbf{u}\cdot\boldsymbol{v}) = \int_0^T \int_{\Gamma^{\boldsymbol{\eta}}(t)} (\partial_t \boldsymbol{\eta}\cdot\mathbf{n}) J^{-1}(\mathbf{u}\cdot\boldsymbol{v}) \\ &= \int_0^T \int_{\Gamma^{\boldsymbol{\eta}}(t)} (\partial_t \boldsymbol{\eta}\cdot\mathbf{n}^{\boldsymbol{\eta}})(\mathbf{u}\cdot\boldsymbol{v}) = \int_0^T \int_{\Gamma^{\boldsymbol{\eta}}(t)} (\mathbf{u}\cdot\mathbf{n}^{\boldsymbol{\eta}})(\mathbf{u}\cdot\boldsymbol{v}),\end{aligned}$$

where $J = \|\mathbf{n}\|$ is the Jacobian of the transformation from Γ to $\Gamma^{\boldsymbol{\eta}}(t)$, and $\mathbf{n}^{\boldsymbol{\eta}}$ is the outer unit normal on $\Gamma^{\boldsymbol{\eta}}(t)$. Thus, we have shown that the limiting functions satisfy the weak form (3.25).

3.9.3 The main result

We have shown that the limiting functions $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ satisfy the weak form of Problem 1 in the sense of Definition 3.1, for all test functions $(\boldsymbol{v}_N, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$. The following theorem holds:

Theorem 3.17 Let $\mathbf{u}_0 \in L^2(\Omega^{\boldsymbol{\eta}}(t))$, $\boldsymbol{\eta}_0 \in H^1(\omega)$, $\mathbf{v}_0 \in L^2(R;\omega)$, $(\mathbf{d}_0, \mathbf{w}_0) \in V_S$, $(\mathbf{k}_0, \mathbf{z}_0) \in L^2(\mathcal{N};\mathbb{R}^6)$ be such that

$$\nabla\cdot\mathbf{u}_0 = 0,\ ((\mathbf{u}_0 \circ \boldsymbol{\phi}^{\boldsymbol{\eta}})|_\Gamma \circ \boldsymbol{\varphi})\cdot\mathbf{e}_r = (\mathbf{v}_0)_r,\ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0,\ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0,$$

and let all the physical constants be positive: $\rho_K, \rho_S, \rho_F, \lambda, \mu, \mu_F > 0$ and $A_i > 0, \forall i = 1,\ldots,n_E$, and let $P_{in/out} \in L^2_{loc}(0,\infty)$. Furthermore, we assume an additional regularity estimate on the approximate shell displacement, i.e. $\|\boldsymbol{\eta}_N\|_{W^{1,\infty}} \le C, \forall t \le T, \forall N \in \mathbb{N}$. Then for every such T there exists a weak solution to Problem 1 in the sense of Definition 3.1.

Remark The additional regularity assumption on the approximate shell displacement that we assumed is not artificial. For example, if we have a structure with a regularization term of sixth order, like in the paper by Boulakia [11] (this term can be physically interpreted as a tripolar material, for example, see [63]), then the elastic operator $\mathcal{L}$ is coercive in H^3, and, by Sobolev embedding, we indeed obtain that the shell displacement is a Lipschitz function.

Remark Let us emphasize one more time the significance of difficulties that arise from the fact that the shell is moving in all three spatial directions. If we had only radial displacement non-negligible, we would not need an additional regularity assumption on the shell displacement. Namely, the Korn's equality for the fluid space would hold true (see [23],[59]), so the convergence of the gradients would be straightforward.

Conclusion

In this thesis we proved the existence of weak solutions to the coupled problem of fluid, mesh and shell interaction, both in the linear and nonlinear, moving-boundary case. The main tools that we employed in order to obtain these results were: Lie operator splitting method together with the semi-discretization, Arbitrary Lagrangian-Eulerian mapping and compactness of sequences in Bochner spaces $L^2(0, T; H(t))$, where $H(t)$ is a family of Hilbert spaces parameterized by t.

In the linear case, where the stationary fluid-structure interaction problem was considered, we used the time-discretization via Lie operator splitting to decouple the problem into two subproblems with different physical properties (fluid and composite structure). We then proved that the solution to the semi-discrete problem converges to a weak solution of the continuous problem, as the time-discretization step tends to zero. We emphasize that the weak convergence was sufficient to be able to pass to the limit.

Regarding the nonlinear problem, we also employed Lie operator splitting scheme to decouple the problem into two subproblems. What we also needed is the Arbitrary-Lagrangian Eulerian mapping to deal with the motion of the fluid boundary since the approximate fluid velocities (obtained by the time-discretization) were defined on different domains. Passing to the limit was not straightforward, since we had to show that the sequences of approximate solutions converge strongly in appropriate function spaces. To obtain strong convergence, we had to employ a recent compactness result obtained by Muha and Čanić in their paper [62].

Appendix

This Appendix gives a brief overview of the notation used throughout this thesis.

The fluid

Domain and mappings	Ω	reference fluid domain
	Γ	reference elastic boundary
	$\Gamma_{in/out}$	inlet and outlet boundary
	$\boldsymbol{\phi}^{\eta}(t)$	deformation of the fluid domain
	$\Omega^{\boldsymbol{\eta}}(t)$	deformed fluid domain
	Γ	deformed elastic boundary
Constants	μ_F	dynamic viscosity
	ρ_F	fluid density
	$P_{in/out}(t)$	inlet and outlet pressure
Unknowns	$\mathbf{u} = (u_z, u_x, u_y)$	velocity
	$\mathbf{D}(\mathbf{u})$	symmetrized gradient
	$\boldsymbol{\sigma}$	Cauchy stress tensor
	p	pressure
	$\boldsymbol{\upsilon}$	associated test functions

The shell

Domain and mappings	$\omega = (0, L) \times (0, 2\pi)$	shell domain
	$\boldsymbol{\varphi}$	shell parameterization
	$\Gamma = \boldsymbol{\varphi}(\omega)$	reference shell domain
Constants	h	thickness
	L	length
	R	reference radius
	ρ_K	shell density
	ε_K	regularization parameter
Unknowns	$\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta)$	displacement
	$\boldsymbol{\varrho}(\boldsymbol{\eta})$	linearized change of metric tensor
	$\boldsymbol{\gamma}(\boldsymbol{\eta})$	linearized change of curvature tensor
	$\boldsymbol{\mathcal{L}}$	elastic operator
	$\mathbf{v} = \partial_t \boldsymbol{\eta}$	velocity of the displacement
	$\boldsymbol{\psi}$	associated test functions

The mesh

Domain and mappings	$I_i = [0, l_i]$	$i-$th rod domain
	$\mathbf{P}_i$	$i-$th rod parameterization on ω
	$\boldsymbol{\pi}_i = \boldsymbol{\varphi}^{-1} \circ \mathbf{P}_i$	$i-$th rod parameterization on Γ
	$\mathcal{N} = \prod_{i=1}^{n_E}(0, l_i)$	mesh net topology
Constants	n_E	number of rods in the mesh
	A_i	area of the cross-section
	M_i	moment of inertia
	H_i	matrix of elastic properties
	Q_i	local basis at each point
	$\mathbf{t}_i$	unit tangent on the middle line
	ρ_S	mesh density
Unknowns	$\mathbf{d}_i$	displacement of the middle line
	$\mathbf{k}_i = \partial_t \mathbf{d}_i$	velocity of the displacement
	$\boldsymbol{\xi}_i$	associated test functions
	$\mathbf{w}_i$	infinitesimal rotation of the cross-section
	$\mathbf{z}_i = \partial_t \mathbf{w}_i$	velocity of the rotation
	$\boldsymbol{\zeta}_i$	associated test functions
	$\mathbf{q}_i, \mathbf{p}_i$	contact moment and force

The function spaces

Basic spaces	$V_F(t)$	fluid space
	V_K	shell space
	V_S	mesh space
	V_{KS}	coupled mesh-shell space
Evolution spaces	$V_F(0,T)$	fluid space
	$V_K(0,T)$	shell space
	$V_S(0,T)$	mesh space
	$V_{KS}(0,T)$	coupled mesh-shell space
Coupled problem spaces	$\mathcal{V}(0,T)$	solution space
	$\mathcal{Q}(0,T)$	test space
Discrete spaces	W_S	structure (mesh-shell) subproblem
	W_F	fluid subproblem

Discretization

Preliminaries	$(0,T)$	time interval
	$\Delta t = T/N$	time-discretization step
	$t_n = n\Delta t$	discrete time for $n = 0, \dots, N$
	$\tilde{\boldsymbol{\eta}}$	reparameterized shell displacement
	$\Omega^n = \boldsymbol{\phi}^{\tilde{\boldsymbol{\eta}}}(n\Delta t, \Omega)$	fluid domain at time t_n
”continuous” ALE mapping	$\mathbf{A}^{\tilde{\boldsymbol{\eta}}}$	ALE mapping from Ω^{n+1} to $\Omega^{\tilde{\boldsymbol{\eta}}}(t)$
	$S^{\tilde{\boldsymbol{\eta}}}$	associated Jacobian
	$\mathbf{s}^{\tilde{\boldsymbol{\eta}}}$	associated domain velocity
”discrete” ALE mapping	$\mathbf{A}^{n+1,n}$	ALE mapping from Ω^{n+1} to Ω^n
	$S^{n+1,n}$	associated Jacobian
	$\mathbf{s}^{n+1,n}$	associated domain velocity
Approximate solutions	$\mathbf{u}_N$	approximate fluid velocity
	$\boldsymbol{\eta}_N$	approximate shell displacement
	$\tilde{\boldsymbol{\eta}}_N$	approximate rep. shell displacement
	$\mathbf{v}_N$	approximate shell velocity
	$\boldsymbol{\eta}_N$	approximate shell displacement
	$\mathbf{d}_N$	approximate mesh displacement
	$\mathbf{w}_N$	approximate mesh rotation
	$\mathbf{k}_N$	approximate mesh velocity
	$\mathbf{z}_N$	approximate mesh rotation velocity

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Curriculum Vitae

Marija Galić was born on 4^{th} of January 1991, in Šibenik, Croatia. She finished her primary and second school in Šibenik, and in September 2009 she started her studies at Department of Mathematics, Faculty of Science, University of Zagreb. She obtained her bachelor's degree in July 2012, and her master's degree in September 2014, with the thesis title *Numerical solutions of Navier-Stokes equations via operator splitting method*, under supervision of prof. Josip Tambača.

She started working as a teaching assistant and enrolled her PhD studies at the same faculty in October 2014, under supervision of prof. Boris Muha. During her PhD studies, she actively participated in numerous international summer/winter schools, workshops and conferences where she held lectures (or presented posters) about her work. Furthermore, from December 2016 until February 2017, she was a PhD intern at the Institute of Mathematics of the Polish Academy of Sciences in Warsaw.

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Her scientific interest includes partial differential equations, numerical analysis and fluid-structure interaction. She is an active member of the *Seminar for differential equations and numerical analysis* at Department of Mathematics, Faculty of Science, University of Zagreb.